PERIPHERAL STRUCTURES OF RELATIVELY HYPERBOLIC GROUPS

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Abstract. In this paper, we introduce and characterize a class of parabolically extended structures for relatively hyperbolic groups. A characterization of relative quasiconvexity with respect to parabolically extended structures is obtained using dynamical methods. Some applications are discussed. The class of groups acting geometrically finitely on Floyd boundaries turns out to be easily understood. However, we also show that Dunwoody's inaccessible group does not act geometrically finitely on its Floyd boundary.

1. Introduction

In this paper, we study peripheral structures of relatively hyperbolic groups. Since introduced by Gromov [18], relative hyperbolicity has several equivalent formulations. If relatively hyperbolic groups are understood as geometrically finite actions(see Bowditch [3]), peripheral structures can be a set of representatives of the conjugacy classes of maximal parabolic subgroups. On the other hand, following approaches of Farb [8] and Osin [28], a peripheral structure is a preferred collection of subgroups such that the constructed relative Cayley graph satisfies some nice properties. In practice a given group may be hyperbolic relative to different peripheral structures. So it is interesting to understand the relationship between possible peripheral structures that can be endowed on a countable group.

The study of peripheral structures actually stems from the study of hyperbolic groups. A first result of this sort is due to Gersten [16] and Bowditch [3], who proved that malnormal quasiconvex subgroups of hyperbolic groups yield peripheral structures. In a point of view of relative hyperbolicity, an ordinary hyperbolic group is hyperbolic relative to a trivial subgroup. Later on, in relatively hyperbolic groups, Osin [29] generalized their results and proposed the notion of hyperbolically embedded subgroups. A hyperbolically embedded subgroup can be added into the existing peripheral structure such that the group is hyperbolic relative to the enlarged peripheral structure. In the present paper, we shall enlarge peripheral subgroups themselves to get new peripheral structures.

In the remainder of the paper, the term "peripheral structure" will be used in a weaker sense, i.e. it is just a finite collection of subgroups without involving relative hyperbolicity.

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Let G be a countable group with a collection of subgroups $\mathbb{H} = \{H_i\}_{i\in I}$. We denote such a pair by $(G, \mathbb{H})$. The collection $\mathbb{H}$ is often referred as a *peripheral structure* of G, and each element of $\mathbb{H}$ a *peripheral subgroup* of G. In what follows, we are interested in peripheral structures of finite cardinality.

Let $\mathbb{H} = \{H_i\}_{i\in I}$ and $\mathbb{P} = \{P_j\}_{j\in J}$ be two peripheral structures of a countable group G. If for each $i \in I$, there exists $j \in J$ such that $H_i \subset P_j$, then we say $\mathbb{P}$ is an *extended peripheral structure* for the pair $(G, \mathbb{H})$. Moreover, if the pairs $(G, \mathbb{H})$ and $(G, \mathbb{P})$ are both relatively hyperbolic, then we say $\mathbb{P}$ is *parabolically extended* for $(G, \mathbb{H})$. Each subgroup $P \in \mathbb{P}$ is said to be *parabolically embedded* into $(G, \mathbb{H})$.

Our first result is to give a characterization of parabolically extended peripheral structure. The notation Γ^g denotes the conjugate $g\Gamma g^{-1}$ of a subgroup $\Gamma \subset G$ by an element $g \in G$. Recall that a subgroup $\Gamma \subset G$ is *weakly malnormal* if $\Gamma \cap \Gamma^g$ is finite for any $g \in G \setminus \Gamma$.

Theorem 1.1. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is an extended peripheral structure for $(G, \mathbb{H})$. Then $(G, \mathbb{P})$ is relatively hyperbolic if and only if each $P \in \mathbb{P}$ satisfies the following statements*

(P1). P is relatively quasiconvex with respect to $\mathbb{H}$,

(P2). P is weakly malnormal, and

(P3). $P^g \cap P'$ is finite for any $g \in G$ and distinct $P, P' \in \mathbb{P}$.

In fact, Theorem 1.1 follows from a characterization of parabolically embedded subgroups (see Theorem 3.10).

Remark 1.2. In our terms, a hyperbolically embedded subgroup $\Gamma \subset G$ in the sense of Osin [29] is parabolically embedded into $(G, \mathbb{H})$. In this case, Γ turns out to be a hyperbolic group. However, parabolically embedded subgroups may be in general hyperbolic relative to a nontrivial collection of proper subgroups, as stated in Condition (P1).

In relatively hyperbolic groups, we can define a natural class of subgroups named relatively quasiconvex subgroups. We observe that a subgroup relatively quasiconvex with respect to one peripheral structure is not necessarily relatively quasiconvex with respect to others. Our second result is to give a characterization of relative quasiconvexity with respect to parabolically extended structures.

Theorem 1.3. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is a parabolically extended structure for $(G, \mathbb{H})$. If $\Gamma \subset G$ is relatively quasiconvex with respect to $\mathbb{H}$, then Γ is relatively quasiconvex with respect to $\mathbb{P}$.*

Conversely, suppose $\Gamma \subset G$ is relatively quasiconvex with respect to $\mathbb{P}$. Then Γ is relatively quasiconvex with respect to $\mathbb{H}$ if and only if $\Gamma \cap P^g$ is relatively quasiconvex with respect to $\mathbb{H}$ for any $g \in G$ and $P \in \mathbb{P}$.

Remark 1.4. Theorem 1.3 generalizes the main result of E. Martinez-Pedroza [23], where $\mathbb{P}$ is obtained from $\mathbb{H}$ by adjoining hyperbolically embedded subgroups.

Unlike the proof of Theorem 1.1, we give a dynamical proof of Theorem 1.3 using the work of Gerasimov [11] and Gerasimov-Potyagailo [13] on Floyd maps.

Using Floyd maps, some preliminary observations are made to general peripheral structures of relatively hyperbolic groups. In this study, it appears worth to distinguish relatively hyperbolic groups which admit geometrically finite actions on their Floyd boundaries and the ones which do not. For instance, peripheral

structures of groups admitting geometrically finite actions on Floyd boundaries are much simpler and shown to be parabolically extended with respect to canonical ones. See Corollary 4.21.

In fact, it is known that many relatively hyperbolic groups act geometrically finitely on their Floyd boundaries:

(1) Geometrically finite Kleinian groups where maximal parabolic subgroups are virtually abelian.
(2) Hyperbolic groups relative to a collection of unconstricted subgroups. According to [6], a group is unconstricted if one of its asymptotic cones has no cut points. By Proposition 4.28 of [27], the Floyd boundary of an unconstricted subgroup is trivial, i.e. consisting of less then 2 points.
(3) Most known hyperbolic groups relative to a collection of non-relatively hyperbolic subgroups. Recall that an *Non-Relatively Hyperbolic* (NRH) group is not hyperbolic relative to any collection of proper subgroups. For example, all NRH subgroups in [1] have trivial Floyd boundaries.

However, there do exist relatively hyperbolic groups which do not act geometrically finitely on their Floyd boundaries. See Proposition 4.23 for the example Dunwoody's inaccessible group which is constructed by Dunwoody in [7].

Moreover, one of our results shows that for a relatively hyperbolic group, its convergence action on Floyd boundary is largely determined by ones of peripheral subgroups.

Theorem 1.5. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic. Then G acts geometrically finitely on its Floyd boundary $\partial_f G$ if and only if each $H \in \mathbb{H}$ acts geometrically finitely on its limit set for the action on $\partial_f G$.*

In [27], Olshanskii-Osin-Sapir made the following conjecture on the relationship between relatively hyperbolic groups and their Floyd boundaries.

Conjecture A. [27] *If a finitely generated group has non-trivial Floyd boundary, then it is hyperbolic relative to a collection of proper subgroups.*

Remark 1.6. The converse of Conjecture A follows from Gerasimov's theorem [10] on Floyd maps for relatively hyperbolic groups.

By simple arguments, we observe that Conjecture A is in fact equivalent to the following cojecture. See Proposition 4.25.

Conjecture B. *If a finitely generated group is hyperbolic relative to a collection of NRH proper subgroups, then it acts geometrically finitely on its Floyd boundary.*

Remark 1.7. One result of Behrstock-Drutu-Mosher [2, Proposition 6.3] says that Dunwoody's inaccessible group does not satisfy the assumption of Conjecture B.

The structure of the paper is as follows. In Section 2, we restate Bounded Coset Penetration property for countable relative hyperbolicity and indicate the equivalence of the definitions of countable relative hyperbolicity due to Osin and Farb. We also construct a quasi-isometric map from a relatively quasiconvex subgroup to the ambient group. Our construction leads to a new proof of Hruska's result [19] that relatively quasiconvex subgroups are relatively hyperbolic. In Section 3, we study parabolically extended structures and give the proof of Theorem 1.1. In Section 4, we take a dynamical approach to relative hyperbolicity and then prove

Theorems 1.3 and 1.5. Dunwoody's inaccessible group and Olshanskii-Osin-Sapir's conjecture are discussed in this section.

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2. Preliminaries

2.1. **Cayley graphs and partial distance functions.** Let G be a group with a set $\mathcal{A} \subset G$. Note that the alphabet set $\mathcal{A}$ is assumed to neither be finite and nor generate G. For convenience, we always assume $1 \notin \mathcal{A}$ and $\mathcal{A} = \mathcal{A}^{-1}$.

We define the *Cayley graph* $\mathscr{G}(G, \mathcal{A})$ of a group G with respect to $\mathcal{A}$, as a directed edge-labeled graph with the vertex set $V(\mathscr{G}(G, \mathcal{A})) = G$ and the edge set $E(\mathscr{G}(G, \mathcal{A})) = G \times \mathcal{A}$. An edge $e = [g, a]$ goes from the vertex g to the vertex ga and has the label $\mathbf{Lab}(e) = a$. As usual, we denote the origin and the terminus of the edge e, i.e., the vertices g and ga, by e_- and e_+ respectively. By definition, we set $e^{-1} := [ga, a^{-1}]$.

Let $p = e_1 e_2 \dots e_k$ be a combinatorial path in the Cayley graph $\mathscr{G}(G, \mathcal{A})$, where $e_1, e_2, \dots, e_k \in E(\mathscr{G}(G, \mathcal{A}))$. The length of p is the number of edges in p, i.e. $\ell(p) = k$. We define the label of p as $\mathbf{Lab}(p) = \mathbf{Lab}(e_1)\mathbf{Lab}(e_2) \dots \mathbf{Lab}(e_k)$. The path p^{-1} is defined in a similar way. We also denote by $p_- = (e_1)_-$ and $p_+ = (e_k)_+$ the origin and the terminus of p respectively. A *cycle* p is a path such that $p_- = p_+$.

Definition 2.1. (Partial Distance Functions) By assigning the length of each edge in $\mathscr{G}(G, \mathcal{A})$ to be 1, we define a *partial distance function* $d_{\mathcal{A}} : \mathscr{G}(G, \mathcal{A}) \times \mathscr{G}(G, \mathcal{A}) \to [0, \infty]$ as follows. Note that $\mathcal{A}$ is not assumed to generate G and thus $\mathscr{G}(G, \mathcal{A})$ may be disconnected. For $z, w \in \mathscr{G}(G, \mathcal{A})$, if z and w lie in the same path connected component of $\mathscr{G}(G, \mathcal{A})$, we define $d_{\mathcal{A}}(z, w)$ as the length of a shortest path in $\mathscr{G}(G, \mathcal{A})$ between z and w. Otherwise we set $d_{\mathcal{A}}(z, w) = \infty$.

Remark 2.2. If $\langle \mathcal{A} \rangle = G$, then the partial distance function $d_{\mathcal{A}}$ actually gives a *word metric* with respect to $\mathcal{A}$ on the Cayley graph $\mathscr{G}(G, \mathcal{A})$. Note that if $g_1, g_2 \in G$ and $g_1^{-1} g_2 \notin \langle \mathcal{A} \rangle$, then $d_{\mathcal{A}}(g_1, g_2) = \infty$. For any element $g \in G$, we define its norm $|g|_{\mathcal{A}} = d_{\mathcal{A}}(1, g)$.

A path p in the Cayley graph $\mathscr{G}(G, \mathcal{A})$ is called (λ, c)-*quasigeodesic* for some $\lambda \geq 1$, $c \geq 0$, if the following inequality holds for any subpath q of p,

$$\ell(p) \leq \lambda d_{\mathcal{A}}(q_-, q_+) + c.$$

We often consider a group G with a collection of subgroups $\mathbb{H} = \{H_i\}_{i \in I}$. Then X is a *relative generating set* for $(G, \mathbb{H})$ if G is generated by the set $(\cup_{i \in I} H_i) \cup X$ in the traditional sense.

Let $\mathcal{H} = \bigsqcup\limits_{i \in I} H_i \setminus \{1\}$. Fixing a relative generating set X for $(G, \mathbb{H})$, the constructed Cayley graph $\mathscr{G}(G, X \cup \mathcal{H})$ is called the *relative Cayley graph* of G with respect to $\mathbb{H}$. We now collect some notions introduced by Osin [28] in relative Cayley graphs.

Definition 2.3. Let p, q be paths in $\mathscr{G}(G, X \cup \mathcal{H})$. A subpath s of p is called an H_i-*component*, if s is the maximal subpath of p such that s is labeled by letters from H_i.

Two H_i-components s, t of p, q respectively are called *connected* if there exists a path c in $\mathscr{G}(G, X \cup \mathcal{H})$ such that $c_- = s_-, c_+ = t_-$ and c is labeled by letters from H_i. An H_i-component s of p is *isolated* if no other H_i-component of p is connected to s.

We say a path p *without backtracking* by meaning that all H_i-components of p are isolated. A vertex u of p is *nonphase* if there is an H_i-component s of p such that u is a vertex of s but $u \neq s_-, u \neq s_+$. Other vertices of p are called *phase*.

2.2. **Relatively hyperbolic groups.** In the large part of this paper, we consider countable relatively hyperbolic groups. In this subsection, we shall recall the definitions of countable relative hyperbolicity in the sense of Osin and Farb, and then indicate their equivalence based on Osin's results in [28].

Let G be a countable group with a finite collection of subgroups $\mathbb{H} = \{H_i\}_{i \in I}$. As the notion of relative generating sets, we can define in a similar fashion the relative presentations and (relative) Dehn functions of G with respect to $\mathbb{H}$. We refer the reader to [28] for precise definitions.

We now give the first definition of relative hyperbolicity due to Osin [28]. Note that the full version of Osin's definition applies to general groups without assuming the finiteness of $\mathbb{H}$.

Definition 2.4. (Osin Definition) A countable group G is *hyperbolic relative to* $\mathbb{H}$ *in the sense of Osin* if G is finitely presented with respect to $\mathbb{H}$ and the relative Dehn function of G with respect to $\mathbb{H}$ is linear.

The following lemma plays an important role in Osin's approach [28] to relative hyperbolicity. The finite subset Ω and constant κ below depend on the choice of finite relative presentations of G with respect to $\mathbb{H}$. In our later use of Lemma 2.5, when saying there exists κ, Ω such that the inequality (1) below holds in $\mathscr{G}(G, X \cup \mathcal{H})$, we have implicitly chosen a finite relative presentation of G with respect to $\mathbb{H}$.

Lemma 2.5. [28, Lemma 2.27] *Suppose $(G, \mathbb{H})$ is relatively hyperbolic in the sense of Osin and X is a finite relative generating set for $(G, \mathbb{H})$. Then there exists $\kappa \geq 1$ and a finite subset $\Omega \subset G$ such that the following holds. Let c be a cycle in $\mathscr{G}(G, X \cup \mathcal{H})$ with a set of isolated H_i-components $S = \{s_1, \ldots, s_k\}$ of c for some $i \in I$, Then*

$$\sum_{s \in S} d_{\Omega_i}(s_-, s_+) \leq \kappa \ell(c), \tag{1}$$

where $\Omega_i := \Omega \cap H_i$.

Remark 2.6. By the definition of d_{Ω_i}, if $d_{\Omega_i}(g, h) < \infty$ for $g, h \in G$, then there exists a path p labeled by letters from Ω_i in this new Cayley graph $\mathscr{G}(G, X \cup \Omega \cup \mathcal{H})$ such that $p_- = g, p_+ = h$.

Using Lemma 2.5, the following lemma can be proven exactly as Proposition 3.15 in [28]. The finite set Ω below is given by Lemma 2.5.

Lemma 2.7. [28] *Suppose $(G, \mathbb{H})$ is relatively hyperbolic in the sense of Osin and X is a finite relative generating set for $(G, \mathbb{H})$. For any $\lambda \geq 1, c \geq 0$, there exists a constant $\epsilon = \epsilon(\lambda, c) > 0$ such that the following holds. Let p, q be (λ, c)-quasigeodesics without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$ such that $p_- = q_-, p_+ = q_+$. Then for any phase vertex u of p(resp. q), there exists a phase vertex v of q(resp.p) such that $d_{X \cup \Omega}(u, v) < \epsilon$.*

The following lemma is well-known in the theory of relatively hyperbolic groups.

Lemma 2.8. [28] *Suppose $(G, \mathbb{H})$ is relatively hyperbolic in the sense of Osin. Then the following statements hold for any $g \in G$ and $H_i, H_j \in \mathbb{H}$,*
1) If $H_i^g \cap H_i$ is infinite, then $g \in H_i$,
2) If $i \neq j$, then $H_i^g \cap H_j$ is finite.

In order to formulate the BCP property, we shall put a metric d_G on a group G, which is *proper* if any bounded set is finite, and *left invariant* if $d_G(gx_1, gx_2) = d_G(x_1, x_2)$ for any $g, x_1, x_2 \in G$. For given $g \in G$, we define the norm $|g|_{d_G}$ with respect to d_G to be the distance $d_G(1, g)$.

Let us now recall the following lemma in Hruska-Wise [20], which justifies the use of proper left invariant metrics on countable groups.

Lemma 2.9. [20] *A group is countable if and only if it admits a proper left invariant metric.*

From now on, we assume that $(G, \mathbb{H})$ has a finite relative generating set X.

In terms of proper left invariant metrics, bounded coset penetration property is formulated as follows.

Definition 2.10. (Bounded coset penetration) Let d_G be some (any) proper, left invariant metric on G. The pair $(G, \mathbb{H})$ is said to satisfy the *bounded coset penetration property with respect to* d_G (or BCP property with respect to d_G for short) if, for any $\lambda \geq 1$, $c \geq 0$, there exists a constant $a = a(\lambda, c, d_G)$ such that the following conditions hold. Let p, q be (λ, c)-quasigeodesics without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$ such that $p_- = q_-$, $p_+ = q_+$.

1) Suppose that s is an H_i-component of p for some $H_i \in \mathbb{H}$, such that $d_G(s_-, s_+) > a$. Then there exists an H_i-component t of q such that t is connected to s.

2) Suppose that s and t are connected H_i-components of p and q respectively, for some $H_i \in \mathbb{H}$. Then $d_G(s_-, t_-) < a$ and $d_G(s_+, t_+) < a$.

Remark 2.11. In [19], Hruska proposed to use the partial distance function d_X (with respect to a finite relative generating set X) instead of d_G in the definition of BCP property, and showed that BCP property is independent of the choice of relative generating sets. However, this is generally not true, due to the following example.

Example 2.12. We take a free product $G = L *_F K$ of two finitely generated groups L and K amalgamated over a nontrivial finite group F, which is known to be hyperbolic relative to $\mathbb{H} = \{L, K\}$ in the sense of Farb and Osin. Take a special relative generating set $X = \emptyset$ and construct the relative Cayley graph $\mathscr{G}(G, X \cup \mathcal{H})$. Since $F = L \cap K$ is nontrivial, we take a nontrivial element $f = f_L = f_K \in F$, where f_L and f_K are the corresponding elements in $L \setminus \{1\}$ and $K \setminus \{1\}$ respectively. Thus, there are two different edges p and q with same endpoints 1 and f such that $\mathbf{Lab}(p) = f_L$ and $\mathbf{Lab}(q) = f_K$ in $\mathscr{G}(G, X \cup \mathcal{H})$. Obviously p and q are geodesics and isolated components. Note that $d_X(p_-, p_+) = \infty$. Hence BCP property is not well-defined with respect to d_X.

This example was also known to other researchers in this field, see Remark 2.15 in a latest version of [23]. Moreover, the idea using proper metrics to define BCP property also appeared independently in Martinez-Pedroza [23]. See Subsection 2.3 in [23].

Remark 2.13. We remark that Hruska's arguments in [19] remain valid with the new definition 2.10 of BCP property. So the main result concerning the equivalence of various definitions of relative hyperbolicity in [19] is still correct. For the convenience of the reader, we will give a direct proof of the equivalence of Osin's and Farb's definitions in the remaining part of this subsection.

The following corollary is immediate by an elementary argument.

Corollary 2.14. *BCP property of* $(G, \mathbb{H})$ *is independent of the choice of left invariant proper metrics.*

In view of Corollary 2.14, we shall not mention explicitly proper left invariant metrics when saying the BCP property of $(G, \mathbb{H})$.

With a little abuse of terminology, we also say $(G, \mathbb{H})$ satisfies BCP property with respect to a partial distance function $d_{\mathcal{A}}$ if, for any $\lambda \geq 1$, $c \geq 0$, there exists a constant $a = a(\lambda, c, d_{\mathcal{A}})$ such that the statements of BCP property 1) and 2) are true for $d_{\mathcal{A}}$.

When proving BCP property, we usually do it with respect to some special partial distance function, as stated in the following corollary.

Corollary 2.15. *Let* $\mathcal{A} \subset G$ *be a finite set. If* $(G, \mathbb{H})$ *satisfies BCP property with respect to* $d_{\mathcal{A}}$*, then so does* $(G, \mathbb{H})$ *with respect to any proper left invariant metric.*

The second definition of relative hyperbolicity is due to Farb [8], which will be used in establishing relative hyperbolicity of groups in Section 3.

Definition 2.16. (Farb Definition) A countable group G is *hyperbolic relative to* $\mathbb{H}$ *in the sense of Farb* if the Caylay graph $\mathscr{G}(G, X \cup \mathcal{H})$ is hyperbolic and the pair $(G, \mathbb{H})$ satisfies the BCP property.

As observed in Example 2.12, BCP property is not well-defined with respect to relative generating sets. But the following lemma states that for a given finite relative generating set, we can always find a finite subset Σ such that $(G, \mathbb{H})$ satisfies BCP property with respect to d_Σ.

Lemma 2.17. *Suppose* $(G, \mathbb{H})$ *is relatively hyperbolic in the sense of Osin and* X *is a finite relative generating set for* $(G, \mathbb{H})$*. Then there exists a finite set* $\Sigma \subset G$ *such that then* $(G, \mathbb{H})$ *satisfies BCP property with respect to* d_Σ*.*

Proof. Let Ω be the finite set given by Lemma 2.5 for $\mathscr{G}(G, X \cup \mathcal{H})$. We take a new finite relative generating set $\hat{X} := X \cup \Omega$. Using Lemma 2.5 again, we obtain a finite set Σ and constant $\mu > 1$ such that the inequality (1) holds in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$.

We now verify BCP property 1). Let p, q be (λ, c)-quasigeodesics without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$. Since $\hat{X}$ is finite, the embedding $\mathscr{G}(G, X \cup \mathcal{H}) \hookrightarrow \mathscr{G}(G, \hat{X} \cup \mathcal{H})$ is a quasi-isometry. Regarded as paths in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, p, q are (λ', c')-quasigeodesics without backtracking in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, for some constants $\lambda' \geq 1, c' \geq 0$ depending on $\hat{X}$.

Let $\epsilon = \epsilon(\lambda, c)$ be the constant given by Lemma 2.7. Set

$$a = \mu(\lambda' + 1)(2\epsilon + 1) + c'\mu.$$

We claim that a is the desired constant for the BCP property of $(G, \mathbb{H})$. If not, we suppose there exists an H_i-component s of p such that $d_\Sigma(s_-, s_+) > a$ and no H_i-component of q is connected to s.

By Lemma 2.7, there exist phase vertices u, v of q such that $d_{X\cup\Omega}(s_-, u) < \epsilon, d_{X\cup\Omega}(s_-, v) < \epsilon$. Thus by regarding p, q as paths in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, there exist paths l and r labeled by letters from Ω such that $l_- = e_-, l_+ = u, r_- = e_+$, and $r_+ = v$. We consider the cycle $c := er[u,v]_q^{-1}l^{-1}$ in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, where $[u,v]_q$ denotes the subpath of q between u and v. Since $[u,v]_q$ is a (λ', c')-quasigeodesic, we compute $\ell(c)$ by the triangle inequality and have

$$\ell(c) \leq (\lambda' + 1)(2\epsilon + 1) + c'.$$

Obviously e is an isolated H_i-component of c. Using Lemma 2.5 for the cycle c in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, we have $d_\Sigma(e_-, e_+) < \mu\ell(c) < a$. This is a contradiction.

Therefore, BCP property 1) is verified with respect to d_Σ. BCP property 2) can be proven in a similar way. □

We conclude this subsection with the following theorem which is proven in [28] for finitely generated relatively hyperbolic groups.

Theorem 2.18. *The pair $(G, \mathbb{H})$ is relatively hyperbolic in the sense of Farb if and only if it is relatively hyperbolic in the sense of Osin.*

Proof. By Corollary 2.15, BCP property of $(G, \mathbb{H})$ follows from Lemma 2.17. The hyperbolicity of relative Cayley graph $\mathscr{G}(G, X \cup \mathcal{H})$ is proven in [28, Corollary 2.54]. Thus, $(G, \mathbb{H})$ is relatively hyperbolic in the sense of Farb.

The sufficient part is proven in the appendix of Osin [28] for finitely generated relatively hyperbolic groups. We remark that the only argument involved to use word metrics with respect to finite generating sets is in the proof of Lemma 6.12 in [28]. But Osin's argument also works for any proper left invariant metric. Hence, Osin's proof is through for the countable case. □

2.3. **Relatively quasiconvex subgroups.** In this subsection, we shall explicitly describe a quasi-isometric map between a relatively quasiconvex subgroup to the ambient relatively hyperbolic group.

The existence of such a quasi-isometric map is first proven in [19, Theorem 10.1], but whose statement or proof does not tell explicitly how a geodesic is mapped. Using an argument of Short on the geometry of relative Cayley graph, we carry on a more careful analysis to construct the quasi-isometric map explicitly.

As a byproduct in the course of the construction, we are able to produce a new proof of the relative hyperbolicity of relatively quasiconvex subgroups. This was an open problem in [28] and is firstly answered by Hruska [19] using different methods. During the preparation of this thesis, E. Martinez-Pedroza and D. Wise [22] gave another elementary and self-contained proof of this result.

Definition 2.19. [19] Suppose $(G, \mathbb{H})$ is relatively hyperbolic and d is some proper left invariant metric on G. A subgroup Γ of G is called *relatively σ-quasiconvex* with respect to $\mathbb{H}$ if there exists a constant $\sigma = \sigma(d) > 0$ such that the following condition holds. Let p be an arbitrary geodesic path in $\mathscr{G}(G, X \cup \mathcal{H})$ such that $p_-, p_+ \in \Gamma$. Then for any vertex $v \in p$, there exists a vertex $w \in \Gamma$ such that $d(u, w) < \sigma$.

Corollary 2.20. [19] *Relative quasiconvexity is independent of the choice of proper left invariant metrics.*

In fact, when proving relative quasiconvexity, we usually verify the relative quasiconvexity with respect to some partial distance function, as indicated in the following corollary. See an application of this corollary in the proof of Proposition 3.3.

Corollary 2.21. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and Γ is a subgroup of G. Let $\mathcal{A} \subset G$ be a finite set and $d_{\mathcal{A}}$ the partial distance function with respect to $\mathcal{A}$. If there exists a constant $\sigma = \sigma(d_{\mathcal{A}}) > 0$ such that for any geodesic p with endpoints at Γ, the vertex set of p lies in σ-neighborhood of Γ with respect to $d_{\mathcal{A}}$. Then Γ is relatively quasiconvex.*

We are going to construct the quasi-isometric map. The relatively finitely generatedness of Γ in Lemma 2.22 is also proved by E. Martinez-Pedroza and D. Wise [22]. In particular, it also follows from a more general result of Gerasimov-Potyagailo [14], which states that 2-cocompact convergence groups are finitely generated relative to a set of maximal parabolic subgroups.

Lemma 2.22. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic. Let $\Gamma < G$ be relatively σ-quasiconvex. Then Γ is finitely generated by a finite subset $Y \subset G$ with respect to a finite collection of subgroups*

$$\mathbb{K} = \{H_i^g \cap \Gamma : |g|_d < \sigma, i \in I, \sharp\, H_i^g \cap \Gamma = \infty\}. \tag{2}$$

Moreover, X can be chosen such that $Y \subset X$ and there is a Γ-equivariant quasi-isometric map $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \to \mathscr{G}(G, X \cup \mathcal{H})$.

Proof. The argument is inspired by the one of [28, Lemma 4.14].

For any $\gamma \in \Gamma$, we take a geodesic p in $\mathscr{G}(G, X \cup \mathcal{H})$ with endpoints 1 and γ. Suppose the length of p is n. Let $g_0 = 1, g_1, \dots, g_n = \gamma$ be the consecutive vertices of p. By the definition of relative quasiconvexity, for each vertex g_i of p, there exists an element γ_i in Γ such that $d(g_i, \gamma_i) < \sigma$.

Denote by x_i the element $\gamma_i^{-1} g_i$, and by e_{i+1} the edge of p going from g_i to g_{i+1}. Obviously we have $\gamma_{i+1} = \gamma_i x_i \mathbf{Lab}(e_{i+1}) x_{i+1}^{-1}$.

Set $\kappa = \max\{|x|_d : x \in X\}$. Then κ is finite, as X is finite. Let $Z_0 = \{\gamma \in \Gamma : |\gamma|_d \leq 2\sigma + \kappa\}$ and $Z_{x,y,i} = \{xhy^{-1} : \ h \in H_i\} \cap \Gamma$. Since the metric d is proper, the set $B_\sigma := \{g \in G : |g|_d \leq \sigma\}$ is finite.

For simplifying notations, we define sets

$$\Pi = \{(x, y, i) : x, y \in B_\sigma, i \in I\}$$

and

$$\Xi = \{(x, y, i) : \sharp\, Z_{x,y,i} = \infty, x, y \in B_\sigma, i \in I\}.$$

If e_{i+1} is an edge labeled by a letter from X, then the element $x_i \mathbf{Lab}(e_{i+1}) x_{i+1}^{-1}$ belongs to Z_0. If e_{i+1} is an edge labeled by a letter from H_k, then $x_i \mathbf{Lab}(e_{i+1}) x_{i+1}^{-1}$ belongs to $Z_{x_i, x_{i+1}, k}$. By the construction, we obtain that the subgroup Γ is also generated by the set

$$Z := Z_0 \cup \left(\cup_{(x,y,i) \in \Pi} Z_{x,y,i} \right).$$

For each $(x, y, i) \in \Pi$, if $Z_{x,y,i}$ is nonempty, then we take an element of the form $x h_i y^{-1} \in Z_{x,y,i}$ for some $h_i \in H_i$. Denote by Z_1 the union of all such elements $\bigcup_{(x,y,i) \in \Pi} x h_i y^{-1}$. Note that $Z_1 \subset Z$. Then we have that Γ is generated by the set

$$\hat{Z} := Y \cup \left(\cup_{(z,z,i) \in \Xi} Z_{z,z,i} \right),$$

where $Y := Z_0 \cup Z_1 \cup \left(\bigcup_{(z,z,i)\in\Pi\setminus\Xi} Z_{z,z,i}\right)$. Indeed, for each triple $(x, y, i) \in \Pi$, we have

$$Z_{x,y,i} = Z_{x,x,i} \cdot xh_iy^{-1}, \text{ where } xh_iy^{-1} \in Z_1.$$

On the other direction, it is obvious that $\hat{Z} \subset Z$.

Let $\hat{X} = X \cup Y \cup B_\sigma$. By the above construction, we define a Γ-equivariant map ϕ from $\mathscr{G}(\Gamma, Z)$ to $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$ as follows. For each vertex $\gamma \in V(\mathscr{G}(\Gamma, Z))$, $\phi(\gamma) = \gamma$. For each edge $[\gamma, s] \in E(\mathscr{G}(\Gamma, Z))$, if $s \in Z_0$, then $\phi([\gamma, s]) = [\gamma, s]$; if $s \in Z_{x,y,i}$ for some $(x, y, i) \in \Xi$, then $s = xty^{-1}$ for some $t \in H_i$ and we set $\phi([\gamma, s]) = [\gamma, x][\gamma x, t][\gamma xt, y^{-1}]$.

For any $\gamma_1, \gamma_2 \in V(\mathscr{G}(\Gamma, Z))$, it is easy to see that $d_{\hat{X}\cup\mathcal{H}}(\gamma_1, \gamma_2) < 3d_Z(\gamma_1, \gamma_2)$. For the other direction, we take a geodesic q in $\mathscr{G}(G, X \cup \mathcal{H})$ with endpoints γ_1, γ_2.

Since $\hat{X}$ is finite, there exist constants $\lambda \geq 1, c \geq 0$ depending only on $\hat{X}$, such that the graph embedding $\mathscr{G}(G, X \cup \mathcal{H}) \hookrightarrow \mathscr{G}(G, \hat{X} \cup \mathcal{H})$ is a G-equivariant (λ, c)-quasi-isometry. Thus, q is a (λ, c)-quasigeodesic in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, i.e.

$$d_{X\cup\mathcal{H}}(\gamma_1, \gamma_2) < \lambda d_{\hat{X}\cup\mathcal{H}}(\gamma_1, \gamma_2) + c.$$

Since q is a geodesic in $\mathscr{G}(G, X\cup\mathcal{H})$ ending at Γ, we can apply the above analysis to q and obtain that $d_Z(\gamma_1, \gamma_2) < d_{X\cup\mathcal{H}}(\gamma_1, \gamma_2)$. Then we have

$$d_Z(\gamma_1, \gamma_2) < \lambda d_{\hat{X}\cup\mathcal{H}}(\gamma_1, \gamma_2) + c.$$

Therefore, ϕ is a Γ-equivariant quasi-isometric map.

We now claim the subgraph embedding $\imath : \mathscr{G}(\Gamma, \hat{Z}) \hookrightarrow \mathscr{G}(\Gamma, Z)$ is a Γ-equivariant $(2, 0)$-quasi-isometry. This is due to the following observation: every element of Z can be expressed as a word of $\hat{Z}$ of length at most 2.

Finally, we obtain a Γ-equivariant quasi-isometric map $\iota := \phi\cdot\imath$ from $\mathscr{G}(\Gamma, Y\cup\mathcal{K})$ to $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$. □

Remark 2.23. Eliminating redundant entries of $\mathbb{K}$ such that all entries of $\mathbb{K}$ are non-conjugate in Γ, we keep the same notation $\mathbb{K}$ for the reduced collection. It is easy to see the construction of the quasi-isometric map $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \to \mathscr{G}(G, X \cup \mathcal{H})$ works for the reduced $\mathbb{K}$.

In the following of this subsection, we assume the Γ-equivariant quasi-isometric map $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \to \mathscr{G}(G, X \cup \mathcal{H})$ is the one constructed in Lemma 2.22. In particular X is the suitable chosen relative generating set such that $Y \subset X$.

Lemma 2.24. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic. Let $\Gamma < G$ be relatively σ-quasiconvex. Then the quasi-isometric map $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \to \mathscr{G}(G, X \cup \mathcal{H})$ sends distinct peripheral $\mathbb{K}$-cosets of Γ to a d-distance σ from distinct peripheral $\mathbb{H}$-cosets of G.*

Proof. Taking into account Lemma 2.22 and Remark 2.23, we suppose all entries of $\mathbb{K}$ are non-conjugate. We continue the notations in the proof of Lemma 2.22.

By the construction of ϕ, we can see the map ϕ sends the subset $gZ_{x,x,i}$ to a uniform d-distance σ from the peripheral coset gxH_i of G for each $(x, x, i) \in \Xi$ and $g \in G$. Here σ is the quasiconvex constant associated to Γ. Observe that $\imath : \mathscr{G}(\Gamma, \hat{Z}) \hookrightarrow \mathscr{G}(\Gamma, Z)$ is an embedding. Therefore, we have the quasi-isometric map $\iota = \phi \cdot \imath$ maps each peripheral $\mathbb{K}$-coset to a uniform distance from a peripheral $\mathbb{H}$-coset.

We now show the "injectivity" of ι on $\mathbb{K}$-cosets. Let $\gamma H_i^g \cap \Gamma$, $\gamma' H_{i'}^{g'} \cap \Gamma$ be distinct peripheral $\mathbb{K}$-cosets of Γ, where $\gamma, \gamma' \in \Gamma$ and $H_i^g \cap \Gamma, H_{i'}^{g'} \cap \Gamma \in \mathbb{K}$.

Using Lemma 2.8, it is easy to deduce that if $\gamma(H_i^g \cap \Gamma)\gamma^{-1} \cap (H_{i'}^{g'} \cap \Gamma)$ is infinite, then $i = i'$ and $\gamma \in H_i^g \cap \Gamma$.

It is seen from the above discussion that there is a uniform constant $\sigma > 0$, such that $\iota(\gamma H_i^g \cap \Gamma) \subset N_\sigma(\gamma g H_i)$ and $\iota(\gamma' H_{i'}^{g'} \cap \Gamma) \subset N_\sigma(\gamma' g' H_{i'})$. It suffices to show that $\gamma g H_i \neq \gamma' g' H_{i'}$.

Without loss of generality, we assume that $i = i'$. Suppose, to the contrary, that $\gamma g H_i = \gamma' g' H_i$. Then we have $\gamma g = \gamma' g' h$ for some $h \in H_i$. It follows that $\gamma g H_i g^{-1} \gamma^{-1} = \gamma' g' H_i g'^{-1} \gamma'^{-1}$. This implies that $H_i^g \cap \Gamma$ is conjugate to $H_i^{g'} \cap \Gamma$ in Γ, i.e. $H_i^g \cap \Gamma = (H_i^{g'} \cap \Gamma)^{\gamma^{-1}\gamma'}$. Since any two entries of $\mathbb{K}$ are non-conjugate in Γ, we have $H_i^g \cap \Gamma = H_i^{g'} \cap \Gamma$. As a consquence, we have $\gamma^{-1}\gamma' \in H_i^g \cap \Gamma$, as $H_i^g \cap \Gamma \in \mathbb{K}$ is infinite. This is a contradiction, since we assumed $\gamma H_i^g \cap \Gamma \neq \gamma' H_{i'}^{g'} \cap \Gamma$.

Therefore, ι sends distinct peripheral $\mathbb{K}$-cosets of Γ to a uniform distance from distinct peripheral $\mathbb{H}$-cosets of G. □

Before proceeding to prove the relative hyperbolicity of relatively quasiconvex subgroups, we need justify the finite collection $\mathbb{K}$ in (2) as a set of representatives of Γ-conjugacy classes of $\hat{\mathbb{K}}$ in (3).

Lemma 2.25. [24] *Suppose $(G, \mathbb{H})$ is relatively hyperbolic. Let $\Gamma < G$ be relatively σ-quasiconvex. Then the following collection of subgroups of Γ*

$$\hat{\mathbb{K}} = \{H_i^g \cap \Gamma : \sharp\, H_i^g \cap \Gamma = \infty, g \in G, i \in I\}. \tag{3}$$

consists of finitely many Γ-conjugacy classes. In particular, $\mathbb{K}$ is a set of representatives of Γ-conjugacy classes of $\hat{\mathbb{K}}$.

Proof. This is proven by adapting an argument of Martinez-Pedroza [24, Proposition 1.5] with our formulation of BCP property 2.10. We refer the reader to [24] for the details. □

We are ready to show the relative hyperbolicity of $(\Gamma, \mathbb{K})$. Using notations in the proof of Lemma 2.22, we recall that $\mathbb{K} = \{Z_{x,x,i} : (x, x, i) \in \Xi\}$.

Lemma 2.26. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic. If $\Gamma < G$ is relatively σ-quasiconvex, then $(\Gamma, \mathbb{K})$ is relatively hyperbolic.*

Proof. Recall that ι is the Γ-equivariant quasi-isometric map from $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$ to $\mathscr{G}(G, X \cup \mathcal{H})$. In particular we assumed $Y \subset X$.

We shall prove the relative hyperbolicity of Γ using Farb's definition. First, it is straightforward to verify that $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$ has the thin-triangle property, using the quasi-isometric map ι and the hyperbolicity of $\mathscr{G}(G, X \cup \mathcal{H})$.

Let d_G be a proper left invariant metric on G. Denote by d_Γ the restriction of d_G on Γ. Obviously d_Γ is a proper left invariant metric on Γ. We are going to verify BCP property 1) with respect to d_Γ, for the pair $(\Gamma, \mathbb{K})$. The verification of BCP property 2) is similar.

Let $[\gamma, s]$ be an edge of $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$, where $s \in Z_{x,x,i}$ for some $(x, x, i) \in \Xi$. By the construction of ι, $[\gamma, s]$ is mapped by ι to the concatenated path $[\gamma, x][\gamma x, t][\gamma z t, x^{-1}]$, which clearly contains an H_i-component $[\gamma x, t]$. Note that $|x|_d \le \sigma$. To simplify notations, we reindex $\mathbb{K} = \{K_j\}_{j \in J}$.

Given $\lambda \geq 1$ and $c \geq 0$, we consider two (λ, c)-quasigeodesics p, q without backtracking in $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$ such that $p_- = q_-$, $p_+ = q_+$. By Lemma 2.24, as p, q are assumed to have no backtracking, the paths $\hat{p} = \iota(p)$, $\hat{q} = \iota(q)$ in $\mathscr{G}(G, X \cup \mathcal{H})$ also have no backtracking. Moreover, for each H_i-component $\hat{s}$ of $\hat{p}$(resp. $\hat{q}$), there is a K_j-component s of p(resp. q) such that $\hat{s} \subset \iota(s)$.

Note that paths $\hat{p}, \hat{q}$ are (λ', c')-quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$ for some $\lambda' \geq 1, c' \geq 1$. By BCP property of $(G, \mathbb{H})$, we have the constant $\hat{a} = a(\lambda', c', d_G)$. Set $a = \hat{a} + 2\sigma$, where σ is the quasiconvex constant of Γ. Let s be a K_j-component of p for some $j \in J$. We claim that if $d_\Gamma(s_-, s_+) > a$, then there is a K_j-component t of q connected to s.

By the property of the map ι, there exists an H_i-component $\hat{s}$ of $\hat{p}$ such that the following hold

$$d_G(\hat{s}_-, \iota(s)_-) \leq \sigma,\ d_G(\hat{s}_+, \iota(s)_+) \leq \sigma.$$

Thus, we have $d_G(\hat{s}_-, \hat{s}_+) > \hat{a}$. Using BCP property 1) of $(G, \mathbb{H})$, there exists an H_i-component $\hat{t}$ of $\hat{q}$, that is connected to $\hat{s}$. By the construction of ι, there is a K_k-component t of q for some $k \in J$ such that $\hat{t} \subset \iota(t)$.

Since $\hat{s}$ and $\hat{t}$ are connected as H_i-components, endpoints of $\hat{s}$ and $\hat{t}$ belong to the same H_i-coset. By Lemma 2.24, it follows that $k = j$. Furthermore, endpoints of s and t must belong to the same K_j-coset. Hence s and t are connected in $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$. Therefore, it is verified that $(\Gamma, \mathbb{K})$ satisfies BCP property 1). $\square$

3. Characterization of parabolically embedded subgroups

Convention 1. *Without loss of generality, peripheral structures considered in this section consist of infinite subgroups. It is easy to see that adding or eliminating finite subgroups in peripheral structures still gives relatively hyperbolic groups.*

3.1. Parabolically embedded subgroups.

Let $\mathbb{H} = \{H_i\}_{i \in I}$ and $\mathbb{P} = \{P_j\}_{j \in J}$ be two peripheral structures of a countable group G. Recall that $\mathbb{P}$ is an *extended peripheral structure* for $(G, \mathbb{H})$, if for each $H_i \in \mathbb{H}$, there exists $P_j \in \mathbb{P}$ such that $H_i \subset P_j$. Given $P \in \mathbb{P}$, we define $\mathbb{H}_P = \{H_i : H_i \subset P, i \in I\}$.

Definition 3.1. Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ an extended peripheral structure for $(G, \mathbb{H})$. If $(G, \mathbb{P})$ is relatively hyperbolic, then $\mathbb{P}$ is called a *parabolically extended* structure for $(G, \mathbb{H})$. Moreover, each $P \in \mathbb{P}$ is said to be *parabolically embedded* into $(G, \mathbb{H})$.

In this subsection, we assume that $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is a parabolically extended structure for $(G, \mathbb{H})$.

Fix a finite relative generating set X for $(G, \mathbb{H})$ and thus $(G, \mathbb{P})$. Since $(G, \mathbb{P})$ is relatively hyperbolic, by Lemma 2.5, we obtain a finite subset Ω and $\kappa \geq 1$ such that the inequality (1) holds in $\mathscr{G}(G, X \cup \mathcal{P})$.

Due to Lemma 2.8 and Convention 1, it is worth to mention that we have $\mathbb{H}_P \cap \mathbb{H}_{P'} = \emptyset$, if P, P' are distinct in $\mathbb{P}$. This implies that each $H \in \mathbb{H}$ belongs to exactly one $P \in \mathbb{P}$.

Since $\mathbb{P}$ is an extended structure for $(G, \mathbb{H})$, then for each $H_i \in \mathbb{H}$, there exists a unique $P_j \in \mathbb{P}$ such that $H_i \hookrightarrow P_j$. By identifying $\mathcal{H} \subset \mathcal{P}$, we regard $\mathscr{G}(G, X \cup \mathcal{H})$ as a subgraph of $\mathscr{G}(G, X \cup \mathcal{P})$.

With a slight abuse of notations, a path p in $\mathscr{G}(G, X \cup \mathcal{H})$ will be often thought of as a path in $\mathscr{G}(G, X \cup \mathcal{P})$. The ambiance will be made clear in the context.

The length $\ell(p)$ of a path p should also be understood in the corresponding relative Cayley graphs, but the values are equal by the natural embedding.

Let $\hat{X} = X \cup \Omega$. We first show parabolically embedded subgroups are relatively finitely generated.

Lemma 3.2. *Let $\Gamma = P_j \in \mathbb{P}$ be parabolically embedded into G. Then $Y := \hat{X} \cap \Gamma$ is a finite relative generating set for the pair $(\Gamma, \mathbb{H}_\Gamma)$.*

Proof. For any $\gamma \in \Gamma$, we take a geodesic p in $\mathscr{G}(G, X \cup \mathcal{H})$ with endpoints 1 and γ. In $\mathscr{G}(G, X \cup \mathcal{P})$, we can connect p_- and p_+ by an edge e, labeled by some letter from Γ, such that $e_- = p_-$ and $e_+ = p_+$. Then the path $c := pe^{-1}$ is a cycle in $\mathscr{G}(G, X \cup \mathcal{P})$. Without loss of generality, we assume e is a Γ-component of c.

The following two cases are examined separately.

Case 1. If there is no Γ-component of p connected to e in $\mathscr{G}(G, X \cup \mathcal{P})$, then e is an isolated component of c. By Lemma 2.5, we have

$$d_{\Omega_j}(e_-, e_+) \leq \kappa \ell(c) \leq \kappa(\ell(p) + 1)$$

where $\Omega_j := \Omega \cap \Gamma$. In particular, there is a path q in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$ labeled by letters from Ω_j, such that $q_- = e_-$, $q_+ = e_+$ and

$$\ell(q) = d_{\Omega_j}(e_-, e_+).$$

Hence, the element γ is a word over the alphabet Ω_j.

Case 2. We suppose that $\{e_1, \ldots, e_i, \ldots, e_n\}$ is the maximal set of Γ-components of p such that each e_i is connected to e. Then p can be decomposed as

$$p = p_1 e_1 \ldots p_i e_i \ldots p_n e_n p_{n+1}. \tag{4}$$

Since e_i is a Γ-component of p, each edge of e_i is labeled by an element in Γ. On the other hand, as a subpath of p, e_i has the label $\mathbf{Lab}(e_i)$ which is a word over $X \cup \mathcal{H}$. Observe that each $H \in \mathbb{H}$ belongs to exactly one $P \in \mathbb{P}$. Thus we obtain that each $\mathbf{Lab}(e_i)$ is a word over $(X \cap \Gamma) \cup \mathbb{H}_\Gamma$

Since the vertex set $\{e_-, (e_1)_-, (e_1)_+, \ldots, (e_n)_-, (e_n)_+, e_+\}$ lies in Γ, we can connect pairs of consequent vertices

$$\{e_-, (e_1)_-\}, \ldots, \{(e_k)_+, (e_{k+1})_-\}, \ldots, \{(e_n)_+, e_+\}$$

by edges $s_0, \ldots, s_k \ldots, s_n$ labeled by letters from Γ respectively. We can get $n+1$ cycles $c_k := p_k s_k^{-1}, 1 \leq k \leq n+1$, such that s_k is an isolated Γ-component of c_k.

As argued in *Case 1* for each cycle c_k, we obtain a path q_k in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$ labeled by letters from Ω_j, such that $(q_k)_- = (e_k)_-$, $(q_k)_+ = (s_k)_+$, $\ell(q_k) = d_{\Omega_j}((s_k)_-, (s_k)_+)$ and the following inequality holds

$$\ell(q_k) \leq \kappa \ell(c_k) \leq \kappa(\ell(p_k) + 1). \tag{5}$$

In particular, we obtain a path $\hat{p}$ in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$ as follows

$$\hat{p} := q_1 e_1 \ldots q_i e_i \ldots q_n e_n q_{n+1} \tag{6}$$

with same endpoints as p. Note that the label $\mathbf{Lab}(\hat{p})$ is a word over the alphabet $(\hat{X} \cap \Gamma) \cup \mathcal{H}_\Gamma$. Therefore, γ is a word over $(\hat{X} \cap \Gamma) \cup \mathcal{H}_\Gamma$. □

Proposition 3.3. *Let $\Gamma = P_j \in \mathbb{P}$ be parabolically embedded into G. Then Γ is relatively quasiconvex with respect to $\mathbb{H}$.*

Proof. Since $\hat{X}$ is a finite relative generating set for $(G, \mathbb{H})$, using Lemma 2.5, we obtain a finite set Σ and constant $\mu > 1$ such that the inequality (1) holds in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$.

Let p be a geodesic in $\mathscr{G}(G, X \cup \mathcal{H})$ such that $p_-, p_+ \in \Gamma$. By Corollary 2.21, it suffices to prove that p lies in a uniform neighborhood of Γ with respect to $d_{\hat{X} \cup \Sigma}$.

By Lemma 3.2, we have a finite relative generating set $Y \subset \hat{X}$ for $(\Gamma, \mathbb{H}_\Gamma)$. Then we have $\mathscr{G}(\Gamma, Y \cup \mathcal{H}_\Gamma) \hookrightarrow \mathscr{G}(G, \hat{X} \cup \mathcal{H})$. Let q be a geodesic in $\mathscr{G}(\Gamma, Y \cup \mathcal{H}_\Gamma)$ such that $q_- = p_-, q_+ = p_+$. We claim that q is a quasigeodesic without backtracking in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$. No backtracking of q is obvious. We will show the quasigeodesicity of q.

We apply the same arguments to p, as *Case 2* in the proof of Lemma 3.2. Precisely, we decompose p as (4) and proceed to obtain the inequality (5) and construct a path $\hat{p}$ as in (6). Observe that $\hat{p}$ has the same endpoints as p, and $\mathbf{Lab}(\hat{p})$ is a word over the alphabet $Y \cup \mathcal{H}_\Gamma$. As $\hat{p}$ can be regarded as path in $\mathscr{G}(\Gamma, Y \cup \mathcal{H}_\Gamma)$, we obtain

$$\ell(q) \leq \ell(\hat{p}). \tag{7}$$

Using the inequality (5), we estimate the length of $\hat{p}$ as follows

$$\begin{aligned} \ell(\hat{p}) &= \textstyle\sum_{1 \leq k \leq n+1} \ell(q_k) + \sum_{1 \leq k \leq n} \ell(e_k) \\ &\leq \textstyle\sum_{1 \leq k \leq n+1} \kappa \ell(p_k) + \sum_{1 \leq k \leq n} \ell(e_k) + (n+1)\kappa \leq 2\kappa \ell(p). \end{aligned} \tag{8}$$

Since $\hat{X}$ is finite, the embedding $\mathscr{G}(G, X \cup \mathcal{H}) \hookrightarrow \mathscr{G}(G, \hat{X} \cup \mathcal{H})$ is a quasi-isometry. Thus there are constants $\lambda \geq 1, c \geq 0$, such that the geodesic p in $\mathscr{G}(G, X \cup \mathcal{H})$ is a (λ, c)-quasigeodesic in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$, i.e.:

$$\ell(p) < \lambda d_{\hat{X} \cup \mathcal{H}}(p_-, p_+) + c. \tag{9}$$

Combining (7), (8) and (9), we have

$$\ell(q) \leq 2\kappa\lambda d_{\hat{X} \cup \mathcal{H}}(q_-, q_+) + 2\kappa c \tag{10}$$

It is easy to see the above estimates (7), (8) and (9) can be applied to arbitrary subpath of q. Thus the same inequality as (10) is obtained for arbitrary subpath of q. This proves our claim that q is a $(2\kappa\lambda, 2\kappa c)$-quasigeodesic without backtracking in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$.

As $\kappa \geq 1$, p is a $(2\kappa\lambda, 2\kappa c)$-quasigeodesic in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$. Hence by Lemma 2.7, there exists a constant $\epsilon = \epsilon(\kappa, \lambda, c)$ such that, for each vertex $v \in p$, there is a phase vertex $u \in q$ such that $d_{\hat{X} \cup \Sigma}(u, v) \leq \epsilon$.

On the other hand, the vertex set of q lies entirely in Γ. Thus p lies in a ϵ-neighborhood of Γ with respect to $d_{\hat{X} \cup \Sigma}$. Therefore, we have proven the relative quasiconvexity of Γ with respect to $\mathbb{H}$. □

Lemma 2.26 and Proposition 3.3 together prove the following.

Corollary 3.4. *A parabolically embedded subgroup Γ is hyperbolic relative to $\mathbb{H}_\Gamma$.*

3.2. **Lifting of quasigeodesics.** In this subsection, we assume that $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ an extended structure for $(G, \mathbb{H})$. The results established in this subsection will be applied in subsection 3.3 to prove Theorem 3.10.

To make our discussion more transparent, we first note the following assumption, on which the notion of a lifting path is defined.

Assumption A. *Each $P_j \in \mathbb{P}$ is relatively quasiconvex with respect to $\mathbb{H}$.*

By Lemma 2.22, we assume that each $P_j \in \mathbb{P}$ is finitely generated by a finite set Y_j with respect to $\mathbb{H}_j := \mathbb{H}_{P_j}$. Without loss of generality, we assume X to be a finite relative generating set for $(G, \mathbb{H})$ such that $Y_j \subset X$ for each $j \in J$. So we can identify the relative Cayley graph $\mathscr{G}(P_j, Y_j \cup \mathcal{H}_j)$ of P_j as a subgraph of $\mathscr{G}(G, X \cup \mathcal{H})$. Thus given any path p of $\mathscr{G}(G, X \cup \mathcal{P})$, we can define the *lifting path* of p in $\mathscr{G}(G, X \cup \mathcal{H})$, by replacing each P_j-component of p by a geodesic segment in $\mathscr{G}(P_j, Y_j \cup \mathcal{H}_j)$ with same endpoints.

Precisely, we express the path p in $\mathscr{G}(G, X \cup \mathcal{P})$ in the following form

$$p = s_0 t_0 \dots s_k t_k \dots s_n t_n, \tag{11}$$

where t_k are P_k-components of p and s_k are labeled by letters from X. It is possible that $P_i = P_j$ for $i \neq j$. We allow s_0 and t_n to be trivial.

Let $\iota_k : \mathscr{G}(P_k, Y_k \cup \mathcal{H}_k) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{H})$ be the graph embedding. For each t_k, we take a geodesic segment $\hat{t}_k$ in $\mathscr{G}(P_k, Y_k \cup \mathcal{H}_k)$ such that $(\hat{t}_k)_- = (t_k)_-$ and $(\hat{t}_k)_+ = (t_k)_+$. Then the following constructed path

$$\hat{p} = s_0 \iota(\hat{t}_0) \dots s_k \iota(\hat{t}_k) \dots s_n \iota(\hat{t}_n) \tag{12}$$

is the *lifting path* of p in $\mathscr{G}(G, X \cup \mathcal{H})$.

The following two lemmas require only Assumption A above.

Lemma 3.5. *Lifting of a path without backtracking in $\mathscr{G}(G, X \cup \mathcal{P})$ has no backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$.*

Proof. We assume the path p and its lifting $\hat{p}$ decompose as (11) and (12) respectively. By way of contradiction, we assume that, for some $H \in \mathbb{H}$, there exist H-components r_1, r_2 of $\hat{p}$, such that r_1, r_2 are connected. Since s_k are labeled by letters from X, we have $r_1 \subset \iota(\hat{t}_i), r_2 \subset \iota(\hat{t}_j)$ for some $0 \leq i, j \leq n$.

Note that $\hat{t}_k$ is a geodesic in $\mathscr{G}(P_k, Y_k \cup \mathcal{H}_k)$, which has no backtracking. Hence, by Assumption A and Lemma 2.24, $\iota(\hat{t}_k)$ has no backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$. It follows that $i \neq j$.

Since p is assumed to have no backtracking, we have that $\iota(\hat{t}_i)$ and $\iota(\hat{t}_j)$ lie entirely in distinct cosets $g_1 P_i$ and $g_2 P_j$ respectively. On the other hand, as r_1 and r_2 are connected H-components, their endpoints lie in the same H-coset. By the assumption that $\mathbb{H}$ consists of infinite subgroups, each $H \in \mathbb{H}$ belongs to exact one subgroup $P \in \mathbb{P}$. Thus it follows that $g_1 P_i$ and $g_2 P_j$ coincide. This leads to a contradiction with non-backtrackingness of p. $\square$

Similar arguments as above allow one to prove the following.

Lemma 3.6. *Suppose p, q are two paths in $\mathscr{G}(G, X \cup \mathcal{P})$ such that there are no connected P_j-components of p, q for any $P_j \in \mathbb{P}$. Then for any $H_i \in \mathbb{H}$, their liftings $\hat{p}, \hat{q}$ have no connected H_i-components.*

To deduce our main result Proposition 3.8, we make another assumption as follows.

Assumption B. *Let X be a finite relative generate set for $(G, \mathbb{H})$. There exists $\kappa \geq 1$ such that for any cycle o in $\mathscr{G}(G, X \cup \mathcal{P})$ with a set of isolated Γ-components $R = \{r_1, \dots, r_k\}$, the following holds*

$$\sum_{r \in R} d_{X \cup \mathcal{H}}(r_-, r_+) \leq \kappa \ell(o).$$

Remark 3.7. Lemma 3.13 below states that Assumption B will be satisfied under the assumptions of Theorem 3.10.

Taking this assumption into account, we have the following.

Proposition 3.8. *Lifting of a quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{P})$ is a quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$.*

Proof. To simplify our proof, we prove the proposition for the lifting of a geodesic p in $\mathscr{G}(G, X \cup \mathcal{P})$. General cases follow from a quasi-modification of the inequality in (13) mentioned below.

We assume the path p and its lifting $\hat{p}$ decompose as (11) and (12) respectively. Since ι_k is an embedding, we will write $\hat{t}_k$ instead of $\iota(\hat{t}_k)$ for simplicity. We shall show the lifting path $\hat{p} = s_0\hat{t}_0 \dots s_k\hat{t}_k \dots s_n\hat{t}_n$ is a quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$.

By Lemma 2.22, we have each $\hat{t}_k$ is a (λ, c)-quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$, where the constants $\lambda \geq 1, c \geq 0$ depend on P_k and X. As $\sharp\ \mathbb{P}$ is finite, λ and c can be made uniform for all $P_k \in \mathbb{P}$.

Let q be a geodesic in $\mathscr{G}(G, X \cup \mathcal{H})$ with same endpoints as $\hat{p}$. Since $\mathscr{G}(G, X \cup \mathcal{H}) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{P})$, it is obvious that

$$\ell(p) \leq \ell(q). \tag{13}$$

We consider the cycle $c := pq^{-1}$ in $\mathscr{G}(G, X \cup \mathcal{P})$. For each t_k, we are going to estimate the length of $\hat{t}_k$ in $\mathscr{G}(G, X \cup \mathcal{H})$.

Case 1. The path t_k is isolated in c. By Assumption B, there exists a constant $\kappa \geq 1$ such that

$$d_{X\cup\mathcal{H}}((\hat{t}_k)_-, (\hat{t}_k)_+) \leq \kappa\ell(c) \leq \kappa(\ell(p) + \ell(q)) \leq 2\kappa(\ell(q)). \tag{14}$$

Case 2. The path t_k is not isolated in c. Then t_k is connected to some P_k-component of q, as p is assumed to have no backtracking in $\mathscr{G}(G, X \cup \mathcal{P})$. Let e_1 and e_2 be the first and last P_k-components of q connected to t_k. Note that e_1 may coincide with e_2.

Since e_1 and e_2 are connected to t_k, we can take two edges u and v labeled by letters from P_k such that

$$u_- = (t_k)_-, u_+ = (e_1)_-, v_- = (t_k)_+, v_+ = (e_2)_+.$$

Then $c_1 := p_1uq_1^{-1}$ and $c_2 := v^{-1}p_2q_2{}^{-1}$ are two cycles in $\mathscr{G}(G, X \cup \mathcal{P})$. By the choice of P_k-components e_1 and e_2, we deduce that u and v are isolated P_k-components of c_1 and c_2 respectively. By Assumption B, we have the following inequalities

$$d_{X\cup\mathcal{H}}(u_-, u_+) \leq \kappa\ell(c_1) \leq \kappa(\ell(p) + \ell(q) + 1) \leq 2\kappa\ell(q) + \kappa, \tag{15}$$

and

$$d_{X\cup\mathcal{H}}(v_-, v_+) \leq \kappa\ell(c_2) \leq \kappa(\ell(p) + \ell(q) + 1) \leq 2\kappa\ell(q) + \kappa. \tag{16}$$

Then it follows from (15) and (16) that

$$\begin{aligned} d_{X\cup\mathcal{H}}((t_k)_-, (t_k)_+) &\leq \ell(q) + d_{X\cup\mathcal{H}}(u_-, u_+) + d_{X\cup\mathcal{H}}(v_-, v_+) \\ &\leq (4\kappa + 1)\ell(q) + 2\kappa. \end{aligned} \tag{17}$$

As $\hat{t}_k$ can be regarded as a (λ, c)-quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$, we estimate the length of $\hat{t}_k$ in $\mathscr{G}(G, X \cup \mathcal{H})$ by taking into account (14) and (17),

$$\begin{aligned} \ell(\hat{t}_k) &\leq \lambda d_{X\cup\mathcal{H}}((t_k)_-, (t_k)_+) + c \\ &\leq \lambda(4\kappa + 1)\ell(q) + 2\lambda\kappa + c. \end{aligned} \tag{18}$$

Finally, we have

$$\begin{aligned} \ell(\hat{p}) &= \textstyle\sum_{0\le k\le n} \ell(s_i) + \sum_{0\le k\le n} \ell(\hat{t}_k) \\ &\le \ell(q) + \ell(q)(\lambda(4\kappa+1)\ell(q) + 2\lambda\kappa + c) \\ &\le \lambda(4\kappa+1)(\ell(q))^2 + (2\lambda\kappa + c + 1)\ell(q). \end{aligned} \tag{19}$$

Similarly, we can apply the above estimates to arbitrary subpath of $\hat{p}$ to obtain the same quadratic bound on its length as (19). It is well-known that in hyperbolic spaces a sub-exponential path is a quasigeodesic, see Bowditch [3, Lemma 5.6] for example. Note that $\mathscr{G}(G, X \cup \mathcal{H})$ is hyperbolic. Hence $\hat{p}$ is a quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$. □

Remark 3.9. In [23], Martinez-Pedroza proves a specical case of Proposition 3.8, where $\mathbb{P}$ is obtained from $\mathbb{H}$ by adding hyperbolically embedded subgroups in the sense of Osin [29].

3.3. **Characterization of parabolically embedded subgroups.** Let $\mathbb{H} = \{H_i\}_{i\in I}$ and $\mathbb{K} \subset \mathbb{H}$ be two peripheral structures of a countable group G. We will show the following characterization of parabolically embedded subgroups.

Theorem 3.10. *Let G be hyperbolic relative to $\mathbb{H}$. Assume that*

(C0). $\Gamma \subset G$ contains $\mathbb{K} \subset \mathbb{H}$,
(C1). Γ is relatively quasiconvex,
(C2). Γ is weakly malnormal,
(C3). $\Gamma^g \cap H_i$ is finite for any $g \in G$ and $H_i \in \mathbb{H} \setminus \mathbb{K}$.

Then G is hyperbolic relative to $\{\Gamma\} \cup \mathbb{H} \setminus \mathbb{K}$.

Putting in another way, Theorem 3.10 implies the following

Corollary 3.11. *Under the assumptions of Theorem 3.10, Γ is a parabolically embedded subgroup of G with respect to $\mathbb{K}$.*

We now prove Theorem 1.1 using Theorem 3.10.

Proof of Theorem 1.1. For the sufficient part, Condition (P1) follows from Proposition 3.3. Since $(G, \mathbb{P})$ is relatively hyperbolic, Conditions (P2) and (P3) are direct consequences of Lemma 2.8.

Let $\mathbb{P} = \{P_1, \ldots, P_j, \ldots, P_n\}$. Recall that $\mathbb{H}_{P_j} = \{H_i : H_i \subset P_j; i \in I\}$. Define peripheral structures

$$\mathbb{P}_k = \{P_1, \ldots, P_k\} \cup \left(\mathbb{H} \setminus \cup_{1\le j\le k} \mathbb{H}_{P_j}\right), 0 \le k \le n.$$

Note that $\mathbb{P}_0 = \mathbb{H}$, $\mathbb{P}_n = \mathbb{P}$. By definition, we have $\mathbb{P}_k$ is an extended structure for $(G, \mathbb{P}_{k-1})$ for each $1 \le k \le n$. In particular, Conditions (P1)-(P3) imply that $P_k \subset G$ satisfies Conditions (C0)-(C3) for $(G, \mathbb{P}_{k-1})$. By repeated applications of Theorem 3.10, we obtain $\mathbb{P}_k$ is parabolically extended for $(G, \mathbb{P}_{k-1})$. Finally, we prove that $\mathbb{P}$ is parabolically extended for $(G, \mathbb{H})$. □

In what follows, we have all assumptions of Theorem 3.10 are satisfied.

Choose a finite relative generating set X for $(G, \mathbb{H})$. Let Ω the finite set obtained by using Lemma 2.5 for $\mathscr{G}(G, X \cup \mathcal{H})$. To simplify notations, we denote $\mathbb{P} = \{\Gamma\} \cup \mathbb{H} \setminus \mathbb{K}$.

Since $\Gamma \subset G$ is assumed to satisfy Conditions (C0)–(C3), by Lemma 2.22 we have Γ is finitely generated by a subset Y with respect to $\mathbb{K}$. Without loss of generality,

we assume $Y \subset X$. So the graph embedding $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{H})$ is a quasi-isometric map.

Note that $\mathbb{P}$ satisfies Assumption A. So given any path p of $\mathscr{G}(G, X \cup \mathcal{H})$, we can define the lifting path $\hat{p}$ in $\mathscr{G}(G, X \cup \mathcal{P})$ as in Subsection 3.2. So we have exactly Lemmas 3.5 and 3.6.

Furthermore, by Lemma 3.13 below, we have Assumption B satisfied in the current setting. So we have the following result by Proposition 3.8.

Proposition 3.12. *Under the assumptions of Theorem 3.10. Lifting of a quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{P})$ is a quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$.*

The following Lemma 3.13 is an analogue of Lemma 2.5, without assuming that $(G, \mathbb{P})$ is relatively hyperbolic. Recall that $\mathbb{P} = \{\Gamma\} \cup \mathbb{H} \setminus \mathbb{K}$.

Lemma 3.13. *Under the assumptions of Theorem 3.10. There exists $\mu \geq 1$ such that for any cycle o in $\mathscr{G}(G, X \cup \mathcal{P})$ with a set of isolated Γ-components $R = \{r_1, \ldots, r_k\}$, the following holds*

$$\sum_{r \in R} d_{X \cup \mathcal{H}}(r_-, r_+) \leq \mu \ell(o).$$

We defer the proof of Lemma 3.13 and now finish the proof of Theorem 3.10 by using Proposition 3.12.

Proof of Theorem 3.10. We shall prove the relative hyperbolicity of $(G, \mathbb{P})$ using Farb's definition.

Let pqr be a geodesic triangle in $\mathscr{G}(G, X \cup \mathcal{P})$. We are going to verify the thinness of pqr. Let $\hat{p}, \hat{q}, \hat{r}$ be lifting of p, q, r in $\mathscr{G}(G, X \cup \mathcal{H})$ respectively. Then by Proposition 3.12, there exists $\lambda \geq 1, c \geq 0$ such that $\hat{p}\hat{q}\hat{r}$ is a (λ, c)-quasigeodesic triangle in $\mathscr{G}(G, X \cup \mathcal{H})$.

Since $(G, \mathbb{H})$ is relatively hyperbolic, then $\hat{p}\hat{q}\hat{r}$ is ν-thin for the constant $\nu > 0$ depending on λ, c. That is to say, the side $\hat{p}$ belongs to a ν-neighborhood of the union $q \cup r$. Since $\mathscr{G}(G, X \cup \mathcal{H}) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{P})$, we have $d_{X \cup \mathcal{P}}(x, y) \leq d_{X \cup \mathcal{H}}(x, y)$ for $x, y \in G$. By the construction of lifting paths, we have the vertex set of triangle pqr is contained in a 1-neighborhood of the one of triangle $\hat{p}\hat{q}\hat{r}$ in $\mathscr{G}(G, X \cup \mathcal{P})$. Then pqr is $(\nu + 1)$-thin in $\mathscr{G}(G, X \cup \mathcal{P})$.

Given any $\lambda \geq 1, c \geq 0$, we take two (λ, c)-quasigeodesics p, q without backtracking in $\mathscr{G}(G, X \cup \mathcal{P})$ with same endpoints. Let $\hat{p}, \hat{q}$ be lifting of p, q in $\mathscr{G}(G, X \cup \mathcal{H})$ respectively. By Proposition 3.12, there exist constants $\lambda' \geq 1, c' \geq 0$, such that $\hat{p}, \hat{q}$ are (λ', c')-quasigeodesic without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$.

Let $\hat{X} = X \cup \Omega$. Using Lemma 2.5 again, we obtain a finite set Σ and constant $\mu > 1$ such that the inequality (1) holds in $\mathscr{G}(G, \hat{X} \cup \mathcal{H})$. Let $\epsilon = \epsilon(\lambda', c')$ the constant given by Lemma 2.7.

Suppose s is a Γ-component of p such that no Γ-component of q is connected to s. To verify BCP property 1), it suffices to bound $d_{\hat{X} \cup \Sigma}(s_-, s_+)$ by a uniform constant using Corollary 2.15. BCP property 2) can be verified in a similar way.

Since endpoints of s belong to the vertex set of $\hat{p}$, by Lemma 2.7, there exists vertices $\hat{u}, \hat{v} \in \hat{q}$ such that

$$d_{\hat{X}}(s_-, \hat{u}) < \epsilon, \ d_{\hat{X}}(s_+, \hat{v}) < \epsilon.$$

If $\hat{u}$ is not a vertex of q, then $\hat{u}$ must belong to a lifting of a Γ-component of q. So we can take a phase vertex $u \in p$ such that $d_{X\cup\mathcal{P}}(u, \hat{u}) \leq 1$. Otherwise, we set $u = \hat{u}$. Similarly, we choose a phase vertex v of q such that $d_{X\cup\mathcal{P}}(v, \hat{v}) \leq 1$. We connect $u, \hat{u}$(resp. $v, \hat{v}$) by a path e_u(resp. e_v), which consists of at most one edge labeled by a letter from Γ. The path e_u is trivial if $u = \hat{u}$.

By regarding p, q as paths in $\mathscr{G}(G, \hat{X}\cup\mathcal{P})$, there exist paths l and r in $\mathscr{G}(G, \hat{X}\cup\mathcal{P})$ labeled by letters from $\hat{X}$, such that $l_- = s_-, l_+ = \hat{u}, r_- = s_+, r_+ = \hat{v}$. Let

$$o = sre_v[u, v]_q^{-1}e_u^{-1}l^{-1}$$

be a cycle in $\mathscr{G}(G, \hat{X}\cup\mathcal{P})$, where $[u, v]_q$ denotes the segment of q between u and v. Since $[u, v]_q$ is a (λ, c)-quasigeodesic in $\mathscr{G}(G, \hat{X}\cup\mathcal{P})$, then by the triangle inequality,

$$\begin{aligned} \ell([u, v]_q) &\leq \lambda d_{X\cup\mathcal{P}}(u, v) + c \\ &\leq \lambda(d_{X\cup\mathcal{P}}(u, \hat{u}) + d_{X\cup\Omega}(\hat{u}, s_-) + 1 + d_{X\cup\mathcal{P}}(s_+, \hat{v}) + d_{X\cup\mathcal{P}}(v, \hat{v})) + c \\ &\leq \lambda(3 + 2\epsilon) + c. \end{aligned}$$

It follows that

$$\begin{aligned} \ell(o) &\leq \ell([u, v]_q) + d_{X\cup\mathcal{P}}(u, \hat{u}) + d_{X\cup\Omega}(\hat{u}, s_-) + 1 + d_{X\cup\mathcal{P}}(s_+, \hat{v}) + d_{X\cup\mathcal{P}}(v, \hat{v}) \\ &\leq (\lambda + 1)(3 + 2\epsilon) + c. \end{aligned}$$

By Lemma 3.13, there exists a constant $\mu \geq 1$ such that

$$d_{X\cup\mathcal{H}}(s_-, s_+) \leq \mu\ell(o) \leq \mu(\lambda + 1)(3 + 2\epsilon) + c\mu. \tag{20}$$

Let $\hat{s}$ be the lifting of the Γ-component s in $\mathscr{G}(G, X\cup\mathcal{H})$. As a subpath of $\hat{p}$, $\hat{s}$ is a (λ', c')-quasigeodesic in $\mathscr{G}(G, X\cup\mathcal{H})$. Then we have $\ell(\hat{s}) \leq \lambda' d_{X\cup\mathcal{H}}(s_-, s_+) + c'$.

We consider the cycle in $\mathscr{G}(G, \hat{X}\cup\mathcal{H})$ as follows

$$\hat{o} := \hat{s}r[\hat{u}, \hat{u}]_{\hat{q}}^{-1}l^{-1},$$

where $[\hat{u}, \hat{v}]_{\hat{q}}$ denotes the subpath of $\hat{q}$ between $\hat{u}$ and $\hat{v}$. As $\hat{q}$ is a (λ', c')-quasigeodesic in $\mathscr{G}(G, X\cup\mathcal{H})$, we have

$$\begin{aligned} \ell([\hat{u}, \hat{v}]_{\hat{q}}) &\leq \lambda' d_{X\cup\mathcal{H}}(\hat{u}, \hat{v}) + c' \\ &\leq \lambda'(d_{X\cup\mathcal{P}}(\hat{u}, s_-) + d_{X\cup\mathcal{H}}(s_-, s_+) + d_{X\cup\mathcal{P}}(s_-, \hat{v})) + c' \\ &\leq \lambda'(2\epsilon + d_{X\cup\mathcal{H}}(s_-, s_+)) + c'. \end{aligned}$$

It follows that

$$\begin{aligned} \ell(\hat{o}) &\leq d_{\hat{X}}(\hat{u}, s_-) + \ell(s) + d_{\hat{X}}(s_+, \hat{v}) + \ell([\hat{u}, \hat{v}]_{\hat{q}}) \\ &< 2\epsilon(\lambda' + 1) + \lambda' d_{X\cup\mathcal{H}}(s_-, s_+) + c'. \end{aligned} \tag{21}$$

It is assumed that no Γ-component of q is connected to the Γ-component s of p. By Lemma 3.6, we obtain that, for any $H_i \in \mathbb{H}$, no H_i-component of $\hat{s}$ is connected to an H_i-component of $\hat{q}$. Moreover, $\hat{s}$ has no backtracking by Lemma 2.24. Hence every H_i-component of $\hat{s}$ is isolated in the cycle $\hat{o}$. Using Lemma 2.5 for $\mathscr{G}(G, \hat{X}\cup\mathcal{H})$, we have

$$d_{\hat{X}\cup\Sigma}(s_-, s_+) < d_{X\cup\mathcal{H}}(s_-, s_+) \cdot \kappa\ell(\hat{o}).$$

Observe that $d_{X\cup\mathcal{H}}(s_-, s_+)$ and $\ell(\hat{o})$ are upper bounded by uniform constants, as shown in 20, 21. Thus the distance $d_{\hat{X}\cup\Sigma}(s_-, s_+)$ is also uniformly upper bounded by a constant. Therefore, we have completed the verification of BCP property 1) for $(G, \mathbb{P})$ $\square$

3.4. **Proof of Lemma 3.13.** Under the assumptions of Theorem 3.10, we now prove Lemma 3.13. Our proof is essentially inspired by Osin's arguments in [29]. In particular, we need the following two Lemmas 3.14 and 3.15 analogous to Lemmas 3.1 and 3.2 in [29] respectively.

Let X be a finite relative generating set for $(G, \mathbb{H})$. Recall that two paths p, q in $\mathscr{G}(G, X \cup \mathcal{H})$ are called *k-connected* for $k \geq 0$, if

$$\max\{d_{X\cup\mathcal{H}}(p_-, q_-), d_{X\cup\mathcal{H}}(p_+, q_+)\} \leq k.$$

Since $(G, \mathbb{H})$ is relatively hyperbolic, we obtain the finite set Ω by using Lemma 2.5 for $\mathscr{G}(G, X \cup \mathcal{H})$.

The following lemma requires only the assumption that $(G, \mathbb{H})$ is relatively hyperbolic. It can be proven by combining the proofs of [28, Proposition 3.15] and [29, Lemma 3.1] partially. The details is left to the interested reader, or see the proof in Appendix A of Thesis [32]. We also remark that a result of the same spirit as Lemma 3.14 was obtained in [5, Proposition 1.11].

Lemma 3.14. *For any $\lambda \geq 1$, $c \geq 0$, there exists $\alpha_1 = \alpha_1(\lambda, c) > 0$ such that, for any $k \geq 0$, there exists $\alpha_2 = \alpha_2(k, \lambda, c) > 0$ satisfying the following condition. Let p, q be two k-connected (λ, c)-quasigeodesics in $\mathscr{G}(G, X \cup \mathcal{H})$. If p has no backtracking and u is a phase vertex on p such that $\min\{d_{X\cup\mathcal{H}}(u, p_-), d_{X\cup\mathcal{H}}(u, p_+)\} > \alpha_2$. Then there exists a phase vertex v on q such that $d_{X\cup\Omega}(u, v) \leq \alpha_1$.*

Using Lemma 3.14, the following lemma, although stated in geometric terms, is a reminiscent of [29, Lemma 3.2] and can be proven along the same line with Conditions (C0)–(C3) on Γ. See a proof in Appendix A of Thesis [32].

Lemma 3.15. *For any $\lambda \geq 1$, $c \geq 0$, $k > 0$, there exists $L = L(\lambda, c, k) > 0$ such that the following holds. Let p, q be k-connected (λ, c)-quasigeodesics without backtracking in $\mathscr{G}(G, X \cup \mathcal{H})$ such that p, q are labeled by letters from $\Gamma \setminus \{1\}$. If $\min\{\ell(p), \ell(q)\} > L$, then p and q as Γ-components are connected in $\mathscr{G}(G, X \cup \mathcal{P})$.*

We define a *geodesic n-polygon P* in a geodesic metric space as a collection of n geodesics $p_1, \ldots, p_n$ such that $(p_i)_+ = (p_{i+1})_-$, where i is taken modulo n. The following lemma follows from the proof of [26, Lemma 25], but is weaker.

Lemma 3.16. [26] *There are constants $\beta_1 = \beta_1(\delta) > 0, \beta_2 = \beta_2(\delta) > 0$ such that the following holds for any geodesic n-polygon P in a δ-hyperbolic space. Suppose the set of all sides of P is divided into three subsets R, S and T with length sums Σ_R, Σ_S and Σ_T respectively. If $\Sigma_R > max\{\beta n, 10^3\Sigma_S\}$ for some $\beta \geq \beta_1$. Then there exist distinct sides $p_i \in R, p_j \in R \cup T$ that contain β_2-connected segments of length greater than $10^{-3}\beta$.*

We are now ready to prove Lemma 3.13. Its proof uses crucially the quasi-isometric map $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{H})$.

Proof of Lemma 3.13. Without loss of generality, we assume the cycle o in $\mathscr{G}(G, X \cup \mathcal{P})$ can be written as the following form

$$o = r_1 s_1 \ldots r_n s_n$$

such that $\{r_1, \ldots, r_n\}$ is the maximal set of Γ-components of o. Note that s_n is not trivial. We assume that $\{r_{n_1}, \ldots r_{n_k}\}$ is a set of isolated Γ-components of o.

Consider the geodesic $2n$-polygon $a_1 b_1 \ldots a_n b_n$, where a_i and b_i are geodesics in $\mathscr{G}(G, X \cup \mathcal{H})$ with the same endpoints as r_i and s_i respectively. We divide the sides

of the 2n-polygon into three disjoint sets. Let $R = \{a_{n_1}, \ldots, a_{n_k}\}$, $S = \{b_1, \ldots, b_n\}$ and $T = \{a_i : a_i \notin R\}$. Set $\Sigma_R = \sum_{i=0}^{n} \ell(a_{n_i})$ and $\Sigma_S = \sum_{i=1}^{k} \ell(b_{n_i})$. Obviously $\Sigma_S \leq \ell(o)$.

Let $\hat{r}_i$ be a geodesic segment in $\mathscr{G}(\Gamma, Y \cup \mathcal{K})$ such that $(\hat{r}_i)_- = (r_i)_-$ and $(\hat{r}_i)_+ = (r_i)_+$. Since the embedding $\iota : \mathscr{G}(\Gamma, Y \cup \mathcal{K}) \hookrightarrow \mathscr{G}(G, X \cup \mathcal{H})$ is quasi-isometric, then $\hat{r}_i$ is a (λ, c)-quasigeodesic in $\mathscr{G}(G, X \cup \mathcal{H})$ for some $\lambda \geq 1, c \geq 0$. Moreover, $\hat{r}_i$ has no backtracking by Lemma 2.24.

Let δ denote the hyperbolicity constant of $\mathscr{G}(G, X \cup \mathcal{H})$. By the stability of quasigeodesics in hyperbolic spaces (see [18] or [17]), there exists a constant $\xi = \xi(\delta, \lambda, c)$ such that $\hat{r}_i$ have a uniform Hausdorff ξ-distance from a_i.

Let $\beta_1 = \beta_1(\delta), \beta_2 = \beta_2(\delta)$ be the constants provided by Lemma 3.16, $L = L(\lambda, c, \beta_2 + 2\xi)$ the constant provided by Lemma 3.15.

It suffices to set $\mu = \max\{\beta_1, 10^3, (L + 2\xi) \cdot 10^3\}$ for showing $\Sigma_R \leq \mu\ell(o)$. Suppose, to the contrary, we have $\Sigma_R > \mu\ell(o)$. This yields

$$\Sigma_R > \mu\ell(o) \geq \max\{\mu\ell(o), 10^3\ell(o) \geq \max\{\mu\ell(o), 10^3\Sigma_S\}.$$

By Lemma 3.16, there are distinct sides $a_j \in R$ and $a_k \in R \cup T$, having β_2-connected segments of length at least $\mu \cdot 10^{-3}$. Therefore, there exist $(\beta_2 + 2\xi)$-connected subsegments $q_1 \subset \hat{r}_j, q_2 \subset \hat{r}_k$ such that

$$\min\{\ell(q_1), \ell(q_2)\} \geq \mu \cdot 10^{-3} - 2\xi \geq L.$$

Since q_1, q_2 are (λ, c)-quasigeodesics labeled by letters from Γ, they are connected by Lemma 3.15. Thus, r_j and r_k are connected. This is a contradiction, since r_j is an isolated Γ-component of the cycle o. □

4. Peripheral structures and Floyd boundary

4.1. **Convergence groups and dynamical quasiconvexity.** Let M be a compact metrizable space. We denote by $\Theta^n M$ the set of subsets of M of cardinality n, equipped with the product topology.

A *convergence group action* is an action of a group G on M such that the induced action of G on the space $\Theta^3 M$ is properly discontinuous. Following Gerasimov [10], a group action of G on M is *2-cocompact* if the quotient space $\Theta^2 M/G$ is compact.

Suppose G has a convergence group action on M. Then M is partitioned into a limit set $\Lambda_M(G)$ and discontinuous domain $M \setminus \Lambda_M(G)$. The *limit set* $\Lambda_M(H)$ of a subgroup $H \subset G$ is the set of limit points, where a *limit point* is an accumulation point of some H-orbit in M. An infinite subgroup $P \subset G$ is a *parabolic subgroup* if the limit set $\Lambda_M(P)$ consists of one point, which is called a *parabolic point.* The stabilizer of a parabolic point is always a (maximal) parabolic group. A parabolic point p with stabilizer $G_p := Stab_G(p)$ is *bounded* if G_p acts cocompactly on $M \setminus \{p\}$. A point $z \in M$ is a *conical* point if there exists a sequence $\{g_i\}$ in G and distinct points $a, b \in M$ such that $g_i(z) \to a$, while for all $q \in M \setminus \{z\}$, we have $g_i(q) \to b$.

Convention 2. *For simplicity, we often denote by $G \curvearrowright M$ a convergence group action of G on a compact metrizable M.*

Let us begin with the following simple observation.

Lemma 4.1. *Suppose a group G admits convergence group actions on compact spaces M and N respectively. If there is a G-equivariant surjective map ϕ from M to N, then for any $H < G$, $\phi(\Lambda_M(H)) = \Lambda_N(H)$.*

Proof. Given $x \in \Lambda_M(H)$, by definition, there exists $z \in M$ and $\{h_n\} \subset H$ such that $h_n(z) \to x$ as $n \to \infty$. Since ϕ is a G-equivariant map, we have $h_n(\phi(z)) = \phi(h_n z) \to \phi(x)$. Then $\phi(x)$ is the limit point of of sequence $\{h_n(\phi(z))\}$ and thus $\phi(x) \in \Lambda_N(H)$.

Conversely, for any $y \in \Lambda_N(H)$, there exists $z \in N$ and $\{h_n\} \subset H$ such that $h_n(z) \to y$. Take $w \in M$ such that $\phi(w) = z$. We have $\phi(h_n w) = h_n \phi(w) = h_n z \to y$. After passage to a subsequence, we assume $x \in \Lambda_M(H)$ to be the limit point of sequence $\{h_n w\}$. Then $\phi(h_n w) \to \phi(x)$ by the continuity of ϕ. It follows that $\phi(x) = y$. Therefore, we obtain $\Lambda_N(H) \subset \phi(\Lambda_M(H))$. □

Remark 4.2. Note that in general $\phi^{-1}(\Lambda_N(H)) = \Lambda_M(H)$ is not true. This is readily seen from Lemma 4.16 below.

Definition 4.3. A subgroup H of a convergence group action $G \curvearrowright M$ is *dynamically quasiconvex* if the following set

$$\{gH \in G/H : g\Lambda_N(H) \cap K \neq \emptyset, g\Lambda_N(H) \cap L \neq \emptyset\}$$

is finite, whenever K and L are disjoint closed subsets of M.

Remark 4.4. The notion of dynamical quasiconvexity was introduced by Bowditch [4] in hyperbolic groups and is proven there to be equivalent to the geometrical quasiconvexity.

In the following lemma, we show that dynamical quasiconvexity is kept under an equivariant quotient.

Lemma 4.5. *Suppose a group G admits convergence group actions on compact spaces M and N respectively. Assume, in addition, that there is a G-equivariant surjective map ϕ from M to N. If $H \subset G$ is dynamically quasiconvex with respect to $G \curvearrowright M$, then it is dynamically quasiconvex with respect to $G \curvearrowright N$.*

Proof. Given any disjoint closed subsets K, L of N, we are going to bound the cardinality of the following set

$$\Theta = \{gH \in G/H : g\Lambda_N(H) \cap K \neq \emptyset, g\Lambda_N(H) \cap L \neq \emptyset\}.$$

Let $K' = \phi^{-1}(K)$ and $L' = \phi^{-1}(L)$. Obviously $K' \cap L' = \emptyset$. For each $gH \in \Theta$, we claim $g\Lambda_M(H) \cap K' \neq \emptyset$. Otherwise, we have then

$$\phi(g\Lambda_M(H)) \cap \phi(K') = g\phi(\Lambda_M(H)) \cap K = \emptyset.$$

By Lemma 4.1, we have $g\Lambda_N(H) \cap K = \emptyset$. This is a contradiction. Hence $g\Lambda_M(H) \cap K' \neq \emptyset$. Similarly, we have $g\Lambda_M(H) \cap L' \neq \emptyset$.

By the dynamical quasiconvexity of H with respect to $G \curvearrowright M$, we have Θ is a finite set. Thus, H is dynamically quasiconvex with respect to $G \curvearrowright N$. □

Definition 4.6. A convergence group action of G on M is *geometrically finite* if every limit point of G in M is either a conical or bounded parabolic.

We now summarize as follows the equivalence of several dynamical formulations of relative hyperbolicity. Theorems 4.7 and 4.9 shall enable us to translate the results established in previous sections in dynamical terms.

Theorem 4.7. [3][10][30][31] *Suppose a finitely generated group G acts on M as a convergence group action. Let $\mathbb{P}$ be a set of representatives of the conjugacy classes of maximal parabolic subgroups. Then the following statements are equivalent:*

(1) The pair $(G, \mathbb{P})$ *is relatively hyperbolic in the sense of Farb,*
(2) $G \curvearrowright M$ *is geometrically finite,*
(3) $G \curvearrowright M$ *is a 2-cocompact convergence group action.*

Remark 4.8. The direction (1) ⇒ (2) is due to Bowditch [3]; (2) ⇒ (1) is proved by Yaman [31]; (2) ⇒ (3) is implied in the work of Tukia [30, Theorem 1 C]; (3) ⇒ (2) is proven in Gerasimov [10] without assuming that G is countable and M metrizable.

In Theorem 4.7, the limit set of G with respect to $G \curvearrowright M$ will be referred as *Bowditch boundary* of the relatively hyperbolic group G. We shall often write it as $T_{\mathbb{P}}$, with reference to a particular peripheral structure $\mathbb{P}$. It is shown in [3] that Bowditch boundary is well-defined up to a G-equivariant homeomorphism.

In different contexts, we can formulate the corresponding notions of relative quasiconvexity, which are proven to be equivalent.

Theorem 4.9. [12] [15] [19] *Suppose a finitely generated group G acts geometrically finitely on M. Let Γ be a subgroup of G. Then the following statements are equivalent:*
(1) Γ *is relatively quasiconvex,*
(2) $\Gamma \curvearrowright \Lambda_M(\Gamma)$ *is geometrically finite,*
(3) Γ *is dynamical quasiconvex with respect to* $G \curvearrowright M$*.*

Remark 4.10. The equivalence (1) ⇔ (2) is proved by Hruska [19] for countable relatively hyperbolic groups; (1) ⇔ (3) is proven in Gerasimov-Potyagailo [12].

Lastly we recall a useful result about peripheral subgroups of finitely generated relatively hyperbolic groups.

Lemma 4.11. [6][28][10][19] *Suppose G is finitely generated and hyperbolic relative to $\mathbb{H}$. Then each $H \in \mathbb{H}$ is undistorted in G. Moreover H is relatively quasiconvex in any relatively hyperbolic* $(G, \mathbb{P})$.

Remark 4.12. The undistortedness of peripheral subgroups are proved by Osin [28], Drutu-Sapir [6] and Gerasimov [10], using quite different methods. The last statement is proved by Hruska [19].

4.2. **Floyd boundary and relative hyperbolicity.** In this subsection, we first briefly recall the work of Gerasimov [11] and Gerasimov-Potyagailo [12] on Floyd maps. Based on their results, the Bowditch boundary with respect to a parabolically extended structure is shown as an equivalent quotient, and then the kernel of such an equivariant map is described.

From now on, unless explicitly stated, G is always assumed to be finitely generated by a fixed finite generating set X.

In [9], Floyd introduced a compact boundary for a finitely generated group G. Let f be a suitable chosen function satisfying Conditions (3)–(4) in [13]. We first rescale the length of each edge e of $\mathscr{G}(G, X)$ by $f(n)$, where n is the word distance of the edge e to $1 \in G$. Then we take length metric on $\mathscr{G}(G, X)$ and get the Cauchy completion $\overline{G}_f$ of $\mathscr{G}(G, X)$. The complete metric ρ on $\overline{G}_f$ is called *Floyd metric.* The completion $\overline{G}_f$ is compact, and the remainder $\overline{G}_f \setminus G$ is defined to be the *Floyd boundary* $\partial_f G$ of G with respect to f.

If $\partial_f G$ consists of 0, 1 or 2 points then it is said to be *trivial.* Otherwise, it is uncountable and is called *nontrivial.* If $\partial_f G$ is nontrivial, then G acts on $\partial_f G$ as a convergence group action, by a result of Karlsson [21].

The following Floyd map theorem due to Gerasimov [11] is key to our study of peripheral structures.

Theorem 4.13. [11] *Suppose $G \curvearrowright M$ is 2-cocompact and M contains at least 3 points. Then there exists a continuous G-equivariant map $\phi : \partial_f G \to M$, where $f(n) = \alpha^n$ for some $\alpha \in]0, 1[$ sufficiently close to 1. Furthermore $\Lambda(G) = \phi(\partial_f G)$.*

The map ϕ given by Theorem 4.13 is called *Floyd map.* According to the discussion in [13], the Floyd map ϕ defines a closed G-invariant equivalent relation $\omega := \{(x, y) : \phi(x) = \phi(y), x, y \in \partial_f G\}$, which induces a *shortcut pseudometric* $\tilde{\rho}$ on $\partial_f G$. This shortcut pseudometric is characterized as the maximal pseudometric, among which vanishes on ω and is less then the Floyd metric ρ. See [13] for more details.

Recall that $T_{\mathbb{H}}$ denotes the Bowditch boundary of $G \curvearrowright M$, where $\mathbb{H}$ is a set of representatives of the conjugacy classes of maximal parabolic subgroups of $G \curvearrowright M$.

Moreover, the push-forward of $\tilde{\rho}$ by ϕ is shown to be a metric on $T_{\mathbb{H}}$ in [11], which is called *shortcut metric* (still denoted by $\tilde{\rho}$). Thus, ϕ is a distance decreasing map from $(\partial_f G, \rho)$ to $(T_{\mathbb{H}}, \tilde{\rho})$:

$$\forall x, y \in \partial_f G : \rho(x, y) \geq \tilde{\rho}(\phi(x), \phi(y)). \tag{22}$$

Convention 3. *Given a subgroup $J \subset G$, we denote by $\Lambda_f(J)$ and $\Lambda_{\mathbb{H}}(J)$ limit sets with respect to $G \curvearrowright \partial_f G$ and $G \curvearrowright T_{\mathbb{H}}$ respectively.*

We now recall the characterization of the "kernel" of Floyd maps given in [13]. Note that a more complete characterization appears in [14], but here we do not need that deeper result.

Theorem 4.14. [13] *Suppose $G \curvearrowright T$ is 2-cocompact. Let $\phi : \partial_f G \to T$ be a G-equivariant map. Then*

$$\phi^{-1}(p) = \Lambda_f(G_p)$$

for any parabolic point $p \in T$. Moreover, the multivalued inverse map φ^{-1} is injective on conical points of $G \curvearrowright T$.

In the following two lemmas, we shall show the Bowditch boundary with respect to an extended parabolically structure can be described in a nice way.

Lemma 4.15. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic. Let $\mathbb{P}$ be a parabolically extended structure for $(G, \mathbb{H})$. Then there exists a G-equivariant surjective map φ such that the following diagram commutes*

$$\begin{tikzcd} \partial_f G \arrow[r, "\phi_1"] \arrow[dr, "\phi_2"'] & T_{\mathbb{H}} \arrow[d, "\varphi"] \\ & T_{\mathbb{P}} \end{tikzcd} \tag{23}$$

where ϕ_1 and ϕ_2 are Floyd maps given by Theorem 4.13. Furthermore, φ is a distance decreasing map with respect to the shortcut metrics $d_{\mathbb{H}}$ and $d_{\mathbb{P}}$.

Proof. The following function φ is well-defined:

$$\forall x \in T_{\mathbb{H}} : \varphi(x) = \phi_2 \phi_1^{-1}(x).$$

It is easy to verify that φ is a G-equivariant continuous map.

We now show the last statement of this lemma. Let ω_1 and ω_2 be G-invariant equivalence relations induced by Floyd map ϕ_1 and ϕ_2 respectively. Observe that $\omega_1 \subset \omega_2$. Thus it follows easily that

$$\forall x, y \in T_{\mathbb{H}} : d_{\mathbb{H}}(x, y) \geq d_{\mathbb{P}}(\varphi(x), \varphi(y)).$$

from the definition of shortcut pseudometrics on $\overline{G}_f$. □

The following lemma follows easily from Theorem 4.14 and describes the kernel of the map φ defined in Lemma 4.15.

Lemma 4.16. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is a parabolically extended structure for $(G, \mathbb{H})$. Let $\varphi : T_{\mathbb{H}} \to T_{\mathbb{P}}$ be the G-equivariant surjective map provided by Lemma 4.15. Then*

$$\varphi^{-1}(p) = \Lambda_{\mathbb{H}}(G_p)$$

for any parabolic point $p \in T_{\mathbb{P}}$. Moreover, the multivalued inverse map φ^{-1} is injective on conical points of $G \curvearrowright T_{\mathbb{P}}$.

Proof. Observe that $\varphi^{-1}(p) = \phi_1 \phi_2^{-1}(p)$ for any $p \in T_{\mathbb{P}}$. Suppose $\varphi(a) = \varphi(b) = p$ for $a, b \in T_{\mathbb{H}}$, i.e. $a, b \in \varphi^{-1}(p)$. If p is conical with respect to $G \curvearrowright T_{\mathbb{P}}$, then $\phi_2^{-1}(p)$ consists of one single point. Thus $a = b$.

If p is bounded parabolic with respect to $G \curvearrowright T_{\mathbb{P}}$, then $\phi_2^{-1}(p) = \Lambda_f(G_p)$ using Theorem 4.14. By Lemma 4.1, we obtain $\varphi^{-1}(p) = \phi_1(\Lambda_f(G_p)) = \Lambda_{\mathbb{H}}(G_p)$. The proof is complete. □

4.3. **Proof of Theorem 1.3.** The proof of Theorem 1.3 is divided into the following two propositions. Taking into account Theorem 4.9, the first proposition follows immediately from Lemmas 4.1 and 4.15.

Proposition 4.17. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is a parabolically extended structure for $(G, \mathbb{H})$. If $\Gamma \subset G$ is relatively quasiconvex in G with respect to $\mathbb{H}$, then Γ is relatively quasiconvex in G with respect to $\mathbb{P}$.*

By Theorem 4.9, the second statement of Theorem 1.3 is restated in the following dynamical terms.

Proposition 4.18. *Suppose $(G, \mathbb{H})$ is relatively hyperbolic and $\mathbb{P}$ is a parabolically extended structure for $(G, \mathbb{H})$. Let $\Gamma \subset G$ acts geometrically finitely on $\Lambda_{\mathbb{P}}(\Gamma)$. Then Γ acts geometrically finitely on $\Lambda_{\mathbb{H}}(\Gamma)$ if and only if $\Gamma \cap P_j^g$ acts geometrically finitely on $\Lambda_{\mathbb{H}}(\Gamma \cap P_j^g)$ for any $j \in J$ and $g \in G$.*

Proof. $\Rightarrow$: By Lemma 3.3, each P_j is relatively quasiconvex with respect to $\mathbb{H}$. It is a well-known fact that the intersection of two relatively quasiconvex subgroups is relatively quasiconvex, see for example, [19] and [24]. Hence we have $\Gamma \cap P_j^g$ is relatively quasiconvex with respect to $\mathbb{H}$, and then acts geometrically finitely on its limit set $\Lambda_{\mathbb{H}}(\Gamma \cap P_j^g)$.

$\Leftarrow$: By Lemma 4.15, the map $\varphi : T_{\mathbb{H}} \to T_{\mathbb{P}}$ is a distance decreasing function with respect to the induced shortcut metrics $d_{\mathbb{H}}$ and $d_{\mathbb{P}}$.

Since Γ is relatively quasiconvex with respect to $\mathbb{P}$, then the following set

$$\{\Gamma \cap P_j^g : \sharp\, \Gamma \cap P_j^g = \infty, g \in G, j \in J\}$$

contains finitely many Γ-conjugacy classes, say $\{Q_1, \ldots, Q_n\}$. By Theorem 4.7, each Q_i acts 2-cocompactly on $\Lambda_{\mathbb{H}}(Q_i)$. We shall show that Γ also acts 2-cocompactly on $\Lambda_{\mathbb{H}}(\Gamma)$.

Since Γ acts 2-cocompactly on $\Lambda_{\mathbb{P}}(\Gamma)$, these exists $\epsilon_0 > 0$ such that for any $(x, y) \in \Theta^2(\Lambda_{\mathbb{P}}(\Gamma))$, there exists $\gamma \in \Gamma$ satisfying $d_{\mathbb{P}}(\gamma x, \gamma y) > \epsilon_0$. Similarly, we have a positive constant $\epsilon_i > 0$ for each $i \in I$ such that for any $(x, y) \in \Theta^2(\Lambda_{\mathbb{H}}(Q_i))$, there exists $\gamma \in Q_i$ satisfying $d_{\mathbb{H}}(\gamma x, \gamma y) > \epsilon_i$.

Let $\epsilon := \min\{\epsilon_0, \min\{\epsilon_i : i \in I\}\}$. We now define a compact $L \subset \Theta^2(\Lambda_{\mathbb{H}}(\Gamma))$ as follows

$$L = \{(x, y) \in \Theta^2(\Lambda_{\mathbb{H}}(\Gamma)) : d_{\mathbb{H}}(x, y) \geq \epsilon\}.$$

Then we claim L is a fundamental domain of Γ on $\Theta^2(\Lambda_{\mathbb{H}}(\Gamma))$.

Given distinct points $p, q \in \Lambda_{\mathbb{H}}(\Gamma)$, we have the following two cases to consider:

Case 1. $\phi(p) \neq \phi(q)$. Then there exists $\gamma_0 \in \Gamma$ such that

$$d_{\mathbb{P}}(\gamma_0(\varphi(p)), \gamma_0(\varphi(q))) = d_{\mathbb{P}}(\varphi(\gamma_0 p), \varphi(\gamma_0 q)) > \epsilon_0 > \epsilon.$$

Since φ is a distance decreasing map, we have $d_{\mathbb{H}}(\gamma_0 p, \gamma_0 q) \geq d_{\mathbb{P}}(\phi(\gamma_0 p), \phi(\gamma_0 q))$. This implies $\gamma_0(p, q) \in L$.

Case 2. $\phi(p) = \phi(q)$. By Lemma 4.16, we have the points p, q lie in the limit set $\Lambda_{\mathbb{H}}(Q_i^{\gamma})$ for some $1 \leq i \leq n, \gamma \in \Gamma$, i.e. $(\gamma^{-1}(p), \gamma^{-1}(q)) \in \Lambda_{\mathbb{H}}(Q_i)$. Then there exists an element γ_i from Q_i such that $d_{\mathbb{H}}(\gamma_i \gamma^{-1}(p), \gamma_i \gamma^{-1}(q)) > \epsilon_i > \epsilon$. This implies that $\gamma_i \gamma^{-1}(p, q) \in L$.

Combining the above two cases, we showed that Γ acts 2-cocompactly ad thus geometrically finitely on $\Lambda_{\mathbb{H}}(\Gamma)$. □

Remark 4.19. Using an argument of [23] with Proposition 3.8, one is able to obtain the full generality of Theorem 1.3 for countable relatively hyperbolic groups. We leave the details to the interested reader.

The proof of Proposition 4.18 also produces the following result.

Theorem 4.20. *(Theorem 1.5) Suppose* $(G, \mathbb{H})$ *is relatively hyperbolic. Then* G *acts geometrically finitely on* $\partial_f G$ *if and only if each* $H \in \mathbb{H}$ *acts geometrically finitely on* $\Lambda_f(H)$.

Proof. $\Rightarrow$: Note that each $H \in \mathbb{H}$ is undistorted in G by Lemma 4.11. Since G acts geometrically finitely on $\partial_f G$, then by Theorem 4.9 each $H \in \mathbb{H}$ acts geometrically finitely on $\Lambda_f(H)$.

$\Leftarrow$: In particular, we use the Floyd map $\phi : \partial_f G \to T_{\mathbb{H}}$ instead of the map φ in the proof of Proposition 4.18. Note that F is also a distance decreasing map with respect to ρ and $d_{\mathbb{H}}$. The other arguments are exactly the same as Proposition 4.18. □

4.4. **Some applications.** In this subsection, we give some preliminary results on general peripheral structures. The first result roughly states that if a finitely generated group acts geometrically finitely on its Floyd boundary, then every peripheral structure to which it may be hyperbolic relative are parabolically extended for a canonical peripheral structure. This is a direct corollary to Theorem 4.13.

Corollary 4.21. *Suppose* G *acts geometrically finitely on* $\partial_f G$ *and* $(G, \mathbb{P})$ *is relatively hyperbolic. Then* $\mathbb{P}$ *is parabolically extended for* $(G, \mathbb{H})$*, where* $\mathbb{H}$ *comprises a suitable choice of representatives of the conjugacy classes of maximal parabolic subgroups with respect to* $G \curvearrowright \partial_f G$*, and possibly a trivial subgroup.*

Proof. Let $\phi : \partial_f G \to T_{\mathbb{P}}$ be the Floyd map given by Theorem 4.13. Let $\tilde{\mathbb{H}}$ be the collection of maximal parabolic subgroups with respect to $G \curvearrowright \partial_f G$.

Claim 1. *For each $H \in \tilde{\mathbb{H}}$, there exists $g \in G$ and $j \in J$ such that $H \subset P_j^g$.*

Proof of Claim 1. As $\Lambda_f(H)$ is a parabolic point, then $\phi(\Lambda_f(H))$ is also fixed by H. Hence $\Lambda_{\mathbb{P}}(H)$ consists of one point or two points. If $\Lambda_{\mathbb{P}}(H)$ is one point, then H contains no hyperbolic elements. By [30, Theorem 3A], the stabilizer of $\Lambda_{\mathbb{P}}(H)$ is a maximal parabolic subgroup for the action $G \curvearrowright T_{\mathbb{P}}$. So the claim is proved in this case.

We now show that $\Lambda_{\mathbb{P}}(H)$ could not consist of two points. Suppose not. Let q be the other point in $\Lambda_{\mathbb{P}}(H)$. Then the preimage $\phi^{-1}(q)$ is H-invariant. Take a point $z \in \phi^{-1}(q)$. As H acts properly discontinuously on $\partial_f G \setminus \{\Lambda_f(H)\}$, then the orbit $H(z)$ should converge to $\Lambda_f(H)$. However, we have $\phi(H(z))$ and $\phi(\Lambda_{\mathbb{P}}(H))$ are distinct points. This contradicts to the continuity of ϕ. □

Let $\mathbb{H}_j$ be a set of representatives of the conjugacy classes of maximal parabolic subgroups with respect to $P_j \curvearrowright \Lambda_f(P_j)$.

Claim 2. *The union $\mathbb{H} := \cup_{j\in J}\mathbb{H}_j$ is a set of representatives of $\tilde{\mathbb{H}}$.*

Proof of Claim 2. By Claim 1, we have that $\mathbb{H}$ contains at least a set of representatives of the conjugacy classes of $\tilde{\mathbb{H}}$. Moreover, it is easy to verify that no two entries of $\mathbb{H}$ is conjugate in G. The claim is thus proved. □

If there exists a parabolic subgroup $P \in \mathbb{P}$ such that P is a hyperbolic group, then we may add the trivial subgroup into $\mathbb{H}$. Then by the choice of $\mathbb{H}$, we have that P is parabolically extended for $(G, \mathbb{H})$. □

In view of Corollary 4.21, Theorem 1.3 gives the following corollary, concerning about "universal" relatively quasiconvex subgroups in certain classes of relatively hyperbolic groups.

Corollary 4.22. *If G acts geometrically finitely on $\partial_f G$ and $(G, \mathbb{P})$ is relatively hyperbolic. Then relatively quasiconvex subgroups of G with respect to $G \curvearrowright \partial_f G$ are relatively quasiconvex with respect to $(G, \mathbb{P})$.*

Moverover, the following conjecture is made by Olshanskii-Osin-Sapir on the relationship between relatively hyperbolic groups and their Floyd boundaries.

Conjecture A. [27] *If a finitely generated group has non-trivial Floyd boundary, then it is hyperbolic relative to a collection of proper subgroups.*

In [2], Behrstock-Drutu-Mosher studied Dunwoody's inaccessible group J which is constructed in [7]. In particular, they proved that there exists no collection $\mathbb{P}$ of NRH proper subgroups such that J is hyperbolic relative to $\mathbb{P}$. Moreover, we have the following observation.

Proposition 4.23. *Dunwoody's group J in* [7] *does not act geometrically finitely on its Floyd boundary.*

Proof. By way of contradiction, we suppose $J \curvearrowright \partial_f J$ is geometrically finite. Let $\mathbb{P}$ be a set of representatives of the conjugacy classes of maximal parabolic subgroups with respect to $J \curvearrowright \partial_f J$. Then the Floyd boundary $\partial_f J$ is same as the Bowditch boundary $T_{\mathbb{P}}$. Moreover, the limit set $\Lambda_f(P)$ of each $P \in \mathbb{P}$ consists of only one point.

By Proposition 6.3 in Behrstock-Drutu-Mosher [2], there exists a subgroup $\Gamma \in \mathbb{P}$ such that Γ is hyperbolic relative to a collection of proper subgroups $\mathbb{K} = \{K_j\}_{j \in J}$. By Corollary 1.14 in [6], we have that J is hyperbolic relative to $\mathbb{H} := \mathbb{K} \cup (\mathbb{P} \setminus \{\Gamma\})$.

By Theorem 4.13, we have a G-equivalent Floyd map $\varphi : T_{\mathbb{P}} = \partial_f J \to T_{\mathbb{H}}$. Note that $\Lambda_f(\Gamma)$ consists of one point. Using Theorem 4.14, we will obtain that φ maps $\Lambda_f(\Gamma)$ to different points $\Lambda_{\mathbb{H}}(K_j)$. This gives a contradiction. Hence, the action $J \curvearrowright \partial_f J$ is not geometrically finite □

Remark 4.24. Note that a more direct proof (without using [6, Corollary 1.14]) follows from [14, Theorem C] and Theorem 4.13. A consequence of [14, Theorem C] says that if $(G, \mathbb{H})$ is relatively hyperbolic, then there is a particular Floyd function f such that for each $H \in \mathbb{H}$, the limit set $\Lambda_f(H)$ is homeomorphic to its Floyd boundary $\partial_f H$. On the other hand, Theorem 4.13 implies that a non-elementary relatively hyperbolic group has a non-trivial Floyd boundary. So if J acts geometrically finitely on its Floyd boundary, then Floyd boundary of every $P \in \mathbb{P}$ consists of one point. As above, by [2, Proposition 6.3], there exists $\Gamma \in \mathbb{P}$ acting non-trivially on a compactum, which contradicts to Theorem 4.13.

Recall that a group H is said *Non-Relatively Hyperbolic* (NRH) if H is not hyperbolic relative to any collection of proper subgroups. As suggested by Theorem 4.23, it seems reasonable to conjecture the following.

Conjecture B. *If a finitely generated group is hyperbolic relative to a collection of NRH proper subgroups, then it acts geometrically finitely on its Floyd boundary.*

As a matter of fact, the converse of Conjecture B is true by Corollary 4.21.

Although Conjectures A and B appear to be different claims, they turn out to be equivalent by the following simple arguments.

Proposition 4.25. *Conjecture A is equivalent to Conjecture B.*

Proof. **Conjecture A implies Conjecture B**: Suppose Conjecture B is false. Then there exists a relatively hyperbolic group G with respect to a collection $\mathbb{H}$ of NRH proper subgroups such that G does not act geometrically finitely on its Floyd boundary. Then by Theorem 1.5, there is a parabolic subgroup $H \in \mathbb{H}$ such that the limit set $\Lambda_f(H)$ is nontrivial and the action $H \curvearrowright \Lambda_f(H)$ is not geometrically finite. By Theorem C in [14], $\Lambda_f(H)$ is homeomorphic to the Floyd boundary of H. Therefore, this contradicts to Conjecture A.

Conjecture B implies Conjecture A: Suppose, to the contrary, that there exists a NRH group Γ with non-trivial Floyd boundary. Then we take a free product $G = \Gamma * F_2$, where F_2 is a free group of rank 2. By Conjecture A, G acts geometrically finitely on $\partial_f G$. By Theorem 1.5, we have that Γ also acts geometrically finitely on $\Lambda_f(\Gamma)$. Using again Theorem C in [14], we obtain that Γ acts geometrically finitely on its non-trivial Floyd boundary. This contradicts to the hypothesis that Γ is a non-relatively hyperbolic group. □

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